Oddtown/Eventown

Eventown: $\mathcal{F} \subseteq [n]$ is an Eventown-family of sets if

- $|F| \equiv 0 \pmod 2$ for all $F \in \mathcal{F}$ and
- $|F_1 \cap F_2| \equiv 0 \pmod 2$ for every $F_1, F_2 \in \mathcal{F}$

How large can $|\mathcal{F}|$ be? As large as $2^{\lfloor n/2 \rfloor}$

Construction. For even n:

$\mathcal{F} = \{F \subseteq [n] : |F \cap \{2i-1, 2i\}|$ is even for all $i \in [\frac{n}{2}]\}$

Oddtown: $\mathcal{F} \subseteq [n]$ is an Oddtown-family of sets if

- $|F| \equiv 1 \pmod 2$ for all $F \in \mathcal{F}$ and
- $|F_1 \cap F_2| \equiv 0 \pmod 2$ for every $F_1 \neq F_2 \in \mathcal{F}$

How large can $|\mathcal{F}|$ be?

Oddtown Theorem The maximum size of an Oddtown-family over $[n]$ is n.

Proof. Let $\mathcal{F} = \{F_1, \ldots, F_m\} \subseteq 2^{[n]}$ be an Oddtown-family.

Let $\mathbf{v_i} \in \{0, 1\}^n$ be the characteristic vector of F_i: j^{th} coordinate is 1 if $j \in F_i$, otherwise 0.

Crucial property: $\mathbf{v_i}^T \mathbf{v_j} = |F_i \cap F_j|$

Claim $\mathbf{v_1}, \ldots, \mathbf{v_n}$ is linearly independent over $\mathbb{F}_2$.

Let $\lambda_1 \mathbf{v_1} + \cdots + \lambda_m \mathbf{v_m} = 0$

Then for every i

$$\begin{aligned} 0 &= (\lambda_1 \mathbf{v_1} + \cdots + \lambda_m \mathbf{v_m})^T \mathbf{v_i} \\ &= \lambda_1 \mathbf{v_1}^T \mathbf{v_i} + \cdots + \lambda_i \mathbf{v_i}^T \mathbf{v_i} + \cdots \lambda_m \mathbf{v_m}^T \mathbf{v_i} \\ &= \lambda_i \end{aligned}$$

Since $\mathbf{v_1}, \ldots \mathbf{v_m}$ are linearly independent vectors in an n-dimensional space, $m \leq n$. $\square$

The Linear Algebra bound

The key to the Oddtown proof is the following simple observation:

$\boxed{\textbf{Linear Algebra bound}}$ If $\mathbf{v_1}, \ldots, \mathbf{v_m}$ are a set of linearly independent vectors belonging to the span of the vectors $\mathbf{u_1}, \ldots \mathbf{u_k}$, then $m \leq k$.

The Gram matrix $M = (m_{ij})$ of a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_m}$ is defined by $m_{ij} = \mathbf{v_i}^T \mathbf{v_j}$.

Proposition Vectors $\mathbf{v_1}, \ldots, \mathbf{v_m} \in \mathbb{F}^n$ are linearly independent iff their Gram matrix over $\mathbb{F}$ is nonsingular.

Proof of Oddtown Thm. The Gram matrix of the characteristic vectors of an Oddtown family over $\mathbb{F}_2$ is the identity matrix. Then Apply Linear Algebra bound.$\square$

Explicit Ramsey graphs I

A clique or an independent set of a graph G is called a *homogenous set*.

A graph is k*-Ramsey* if it does not contain a homogenous set of order k. (Remark: instead of RED/BLUE-coloring we formulate in terms of edge/non-edge.)

In a week we will know that the largest k-Ramsey graph has at least $\sqrt{2}^k$ vertices. ($R(k,k) \geq \sqrt{2}^k$.) We will prove its *existence* by the probabilistic method.

BUT: can you give one such beast in my hand?

Sure: go over all the $2^{\binom{n}{2}}$ graphs on $n = \sqrt{2}^k$ vertices and check whether their clique number and independence number are below k (Never mind that these are NP-hard problems).
Eventually you'll find a k-Ramsey graph.

Explicit Ramsey graphs II

Why are you not happy with this “construction”?

It takes too much time.

What is then a constructive k-Ramsey graph?

Its adjacency matrix should be constructible in time polynomial in its number of vertices n.
Or even stronger: adjacency of any two vertices should be decidable in time polynomial in $\log n$ (what it takes to write down the label of the two vertices).

Turán construction: $T_{(k-1)^2,k-1}$ has no clique and no independent set of order k.
Has $(k-1)^2$ vertices.

Anything more than quadratic?

Construction of Nagy

$V(G) = \binom{[k]}{3}$,
$E(G) = \{AB : |A \cap B| = 1\}$

For $k = 3 \rightsquigarrow$ Petersen graph

Theorem G has no homogenous set of order $k + 1$.

Proof. An independent set of G is an Oddtown family $\Rightarrow \ \alpha(G) \leq k$.

For the vertices $C_1, \ldots, C_m$ of a clique, we have that

- $|C_i| = 3$ for all i and
- $|C_i \cap C_j| = 1$ for every $i \neq j$

Hence the Gram matrix of the characteristic vectors is $J_m + 2I_m$ (J_m is the all 1 matrix, I_m is the identity)

M is nonsingular over $\mathbb{R} \Rightarrow \mathbf{v_1}, \ldots, \mathbf{v_m} \in R^n$ are linearly independent $\Rightarrow m \leq n$.

So $\omega(G) \leq n$. $\square$

G is a k-Ramsey graph on $\Theta(k^3)$ vertices.

HW: Linear Algebra-free proof (like original)

Non-uniform Fischer Inequality

Non-uniform Fischer Inequality Let $\lambda \in I\!N$. If $\mathcal{F} = \{F_1, \ldots, F_m\} \subseteq 2^{[n]}$ is a family of sets with $|C_i \cap C_j| = \lambda$ for all $i \neq j$, then

$$|\mathcal{F}| \leq n.$$

Proof: Case 1 $\exists i$ such that $|F_i| = \lambda$.

Consider $\{F_j \setminus F_i : j \neq i\}$ and use induction on n.

Case 2: $\forall i$, we have $|F_i| > \lambda$.

The Gram matrix of $\mathcal{F}$ is $\lambda J_m + D$, where D is the diagonal matrix with diagonal entry $d_i = |F_i| - \lambda > 0$, $i = 1, \ldots, m$.

$\mathbf{x}^T(\lambda J + D)\mathbf{x} = \lambda \left(\sum_{i=1}^m \mathbf{x}_i^2 \right) + \sum_{i=1}^m d_i \mathbf{x}_i^2 > 0$ for every $\mathbf{x} \neq 0$, so the Gram matrix is positive definite.

$\Rightarrow$ the Gram matrix has full rank (m) over $I\!R$.

$\Rightarrow$ the charactersitic vectors are linearly independent

$\Rightarrow m \leq n$.

Construction of Frankl-Wilson

$V(G) = \binom{[n]}{p^2-1}$ with $n = p^3 - 1$

$E(G) = \{AB : |A \cap B| \equiv -1 \pmod{p}\}$

Remark $p = 2 \rightsquigarrow$ Nagy graph

Theorem (Frankl-Wilson) The graph G above gives a constructive k-Ramsey with a vertex set size of order

$$k^{O\left(\frac{\ln k}{\ln \ln k}\right)}.$$

Proof: Next semester in DM III.

Chromatic number of the unit-distance graph

G_n is the n-dimensional unit distance graph.

$V(G_n) = \mathbb{R}^n$
$E(G_n) = \{\mathbf{xy} : \|\mathbf{x} - \mathbf{y}\| = 1\}$

\$1000 dollar question: What is the chromatic number of the plane?
We know $4 \leq \chi(G_2) \leq 7$. (HW)

Hadwiger-Nelson problem How fast does $\chi(G_n)$ grow?

Claim $\chi(G_n) \leq n^{n/2}$. (HW)
$\chi(G_n) \geq n + 1$. (simplex with unit sidelength)

Larman-Rogers (1972) $\chi(G_n) \leq const^n$ (HW)
$\chi(G_n) = \Omega(n^2)$

Remark. Clearly, unit-distance plays no special role here. $G_n \cong G_n^\delta$ where G_n^δ is the "δ-distance graph":
$V(G_n^\delta) = \mathbb{R}^n$
$E(G_n^\delta) = \{\mathbf{xy} : \|\mathbf{x} - \mathbf{y}\| = \delta\}$

The growth of $\chi(G_n)$ is exponential

Theorem (Frankl-Wilson, 1981) $\chi(G_n) \geq \Omega(1.1^n)$.

Proof. **Goal**: For some distance $\delta > 0$ we find a subgraph $H_n \subseteq G_n^\delta$ with $\alpha(H) \leq \frac{|V(H)|}{1.1^n}$.

Key: If $\mathbf{v}_A$ and $\mathbf{v}_B \in \mathbb{R}^n$ are the characteristic vectors of sets A and $B \in 2^{[n]}$, then distance of $\mathbf{v}_A$ and $\mathbf{v}_B$ is equal to $\sqrt{|A \triangle B|}$.

If $A, B \in \mathcal{F} \subseteq \binom{[d]}{k}$ are members of a **uniform** family $\mathcal{F}$, then the distance of $\mathbf{v}_A$ and $\mathbf{v}_B$ depends on the intersection size: $\|\mathbf{v}_A - \mathbf{v}_B\| = \sqrt{2(k - |A \cap B|)}$.

an independent set in G_n^δ avoids distance $\delta \rightsquigarrow$
a uniform family avoiding a certain intersection size.

We give a family $\mathcal{F}$ where any subfamily $\mathcal{F}' \subseteq \mathcal{F}$, whose members **avoid** a certain intersection size, is small compared to $|\mathcal{F}|$:

Let $\mathcal{F} := \binom{[4p-1]}{2p-1}$, where p is a prime.
Then pairwise intersection size $p-1$ is hard to avoid.

Generalized Oddtown

Replacing $\mathbb{F}_2$ in the Oddtown proof with $\mathbb{F}_p$ in the proof immediately gives the

$\mathbf{modp}$-town Theorem. Let p be a prime number and $\mathcal{F} \subseteq 2^{[n]}$ be a family such that

- $|F| \not\equiv 0 \pmod{p}$ for all $F \in \mathcal{F}$ and
- $|F_1 \cap F_2| \equiv 0 \pmod{p}$ for every $F_1 \neq F_2 \in \mathcal{F}$

Then $|\mathcal{F}| \leq n$.

Even More Generalized Oddtown

Theorem ("Nonuniform modular RW-Theorem", Frankl-Wilson, 1981; Deza-Frankl-Singhi, 1983)
Let p be a prime, and L be a set of s integers.
Let $B_1, \ldots, B_m \in 2^{[n]}$ be a family such that

- $|B_i| \notin L \pmod{p}$
- $|B_i \cap B_j| \in L \pmod{p}$ for every $i \neq j$.

Then

$$m \leq \sum_{i=0}^{s} \binom{n}{i}.$$

Remark RW stands for Ray-Chaudhuri and Wilson.

Remark Oddtown Theorem: $p = 2$, $L = \{0\}$.
The statement only gives $m \leq n + 1$, but the proof will give $m \leq n$ (because $L = \{0\}$)

Generalizing linear independence of vectors

Let $\mathbb{F}$ be a field and Ω an arbitrary set.Then the set $\mathbb{F}^{\Omega} = \{f : \Omega \to \mathbb{F}\}$ of functions is a *vector space over* $\mathbb{F}$.

Lemma Let $\Omega \subseteq \mathbb{F}^n$. If $f_1, \ldots, f_m \in \mathbb{F}^{\Omega}$ and there exist $\mathbf{v_1}, \ldots, \mathbf{v_m} \in \Omega$ such that

- $f_i(\mathbf{v_i}) \neq 0$, and
- $f_i(\mathbf{v_j}) = 0$ for all $j < i$,

then $f_1, \ldots, f_m$ are linearly independent in $\mathbb{F}^{\Omega}$.

Proof. Suppose $\lambda_1 f_1 + \cdots + \lambda_m f_m = 0$, and let j be the smallest index with $\lambda_j \neq 0$. Substituting $\mathbf{v_j}$ into this function equation we have

$$\underbrace{\lambda_1 f_1(\mathbf{v_j}) + \cdots + \lambda_{j-1} f_{j-1}(\mathbf{v_j})}_{=0,\text{ since } \lambda_i = 0,\ i<j} + \underbrace{\lambda_j f_j(\mathbf{v_j})}_{\neq 0}$$
$$+ \underbrace{\lambda_{j+1} f_{j+1}(\mathbf{v_j}) + \cdots + \lambda_m f_m(\mathbf{v_j})}_{=0,\text{ since } f_i(\mathbf{v_j}) = 0,\ j<i} = 0,$$

a contradiction. $\square$

Proof of Even More Generalized Oddtown

For each set B_i, let $\mathbf{v_i} \in \mathbb{F}_p$ be its characteristic vector. For $\mathbf{x} = (x_1, \ldots, x_n)$ let

$$f_i(\mathbf{x}) = \prod_{l \in L} (\mathbf{x}^T \mathbf{v_i} - l).$$

Clearly,

$$f_i(\mathbf{v_j}) \begin{cases} \neq 0 & \text{if } i = j \\ = 0 & \text{if } i \neq j \end{cases}$$

So the functions $f_1, \ldots, f_m$ are linearly independent in the subspace they generate in $\mathbb{F}_p[x_1, \ldots, x_m]$. What is the dimension?

Each f_i is the product of s linear functions in n variables. Expanding the parenthesis: f_i is the linear combination of terms of the form $x_1^{s_1} \cdot \cdots \cdot x_n^{s_n}$ with $s_1 + \cdots + s_n = s$?
How many terms like that are there?

Much more than we can afford ...

Multilinearization

We need another trick to reduce the dimension. We use that our vectors (witnessing the linear independence in the Lemma) have only 0 or 1 coordinates.

From f_i define $\tilde{f}_i$ by expanding the product and replacing each power x_i^k by a term x_i for every $k \geq 1$ and i, $1 \leq i \leq m$.

Since $0^k = 0$ and $1^k = 1$ for every $k \geq 1$ we have that $f_i(\mathbf{v_j}) = \tilde{f}_i(\mathbf{v_j})$ for every i, j.

The properties of the functions and vectors remains valid, so the (now) multi**linear** polynomials $\tilde{f}_1, \ldots, \tilde{f}_m$ of total degree s are also linearly independent.

They live in a space spanned by the basic monomials $\prod_{j=1}^{k} x_{i_j}$ of degree at most s. Their number is at most

$$\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{1} + \binom{n}{0}. \qquad \square$$

Avoiding a certain intersection size

Theorem Let p be a prime number. If $\mathcal{F}' \subseteq \binom{[4p-1]}{2p-1}$ such that for all $A, B \in \mathcal{F}'$ we have $|A \cap B| \neq p-1$, then

$$|\mathcal{F}| \leq 2 \cdot \binom{4p-1}{p-1} < 1.76^n.$$

Proof. Consequence of Generalized Oddtown with $L = \{0, 1, 2, \ldots p-2\}$. $\square$

Let $n = 4p-1$, $k = 2p-1$.
Let $\delta = \sqrt{2(2p-1-(p-1))} = \sqrt{2p}$ and define $H \subseteq G_d^\delta$ by $V(H) = \{\mathbf{v}_A : A \in \binom{[n]}{k}\}$.
Then distance δ is hard to avoid in $V(H)$:

$$\alpha(H) \leq 1.76^n < \frac{\binom{4p-1}{2p-1}}{1.1^n}.$$

$\square$

Remark Optimizing parameters gives $\chi(G_n) \geq \Omega(1.2^n)$.

Proof of Eventown bound

Let $\{F_1, \ldots, F_m\}$ be an Eventown family.

Let $V_{\mathcal{F}} = \langle \mathbf{v}_1, \ldots, \mathbf{v}_m \rangle \leq \mathbb{F}_2^n$ be the linear space spanned by the characteristic vectors.

Let $\mathbf{u}_1, \ldots, \mathbf{u}_k \in V_{\mathcal{F}}$ be a basis of $V_{\mathcal{F}}$.

Let $U : \mathbb{F}_2^n \to \mathbb{F}_2^k$ be the linear function defined by $U(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{u}_1, \ldots, \mathbf{x} \cdot \mathbf{u}_k)$.

$\mathbf{u}_1, \ldots, \mathbf{u}_k$ linearly independent, so $\dim im(U) = k$.

Claim $V_{\mathcal{F}} \subseteq ker(U)$

Proof. Any $\mathbf{x} \in V_{\mathcal{F}}$ is a linear combination of $\mathbf{v}_i$s, so is any $\mathbf{u}_i$. So

$$\mathbf{x} \cdot \mathbf{u}_i = \sum_{j,k} \alpha_j \beta_k \mathbf{v}_j \cdot \mathbf{v}_k = 0,$$

since by the Eventown rules $\Rightarrow \mathbf{v}_j \cdot \mathbf{v}_l = 0$ for every $1 \leq j, l \leq m$ □

$k = \dim(V_{\mathcal{F}}) \leq \dim ker(U) = n - k$

$\dim(V_{\mathcal{F}}) \leq \lfloor \frac{n}{2} \rfloor \;\Rightarrow\; m \leq |V_{\mathcal{F}}| \leq 2^{\lfloor n/2 \rfloor}$. □