List Coloring

$v \in V(G)$, $L(v)$ a list of colors
A list coloring is a proper coloring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$.

G is k-choosable or k-list-colorable if **every** assignment of k-element lists permits a proper coloring.

$\chi_l(G) = \min\{k : G \text{ is } k\text{-choosable}\}$

Claim $\chi_l(G) \geq \chi(G)$

Claim $\chi_l(G) \leq \Delta(G) + 1$

Example: K_n, $K_{2,2}$

Example: $\chi_l(K_{3,3}) \neq \chi(K_{3,3})$

Example: $\chi_l(G) - \chi(G)$ arbitrary large

Proposition $K_{m,m}$ is not k-choosable for $m = \binom{2k-1}{k}$.

Edge-List Coloring

List Coloring Conjecture (1985) $\chi'_l(G) = \chi'(G)$

Theorem (Galvin, 1995) $\chi'_l(B) = \chi'(B)$ for any bipartite graph B.

Proof for $B = K_{n,n}$ (Dinitz Conjecture, 1979)

A kernel of a digraph D is an independent set $S \subseteq V(D)$, such that for every $v \in V(D) \setminus S$ there is $w \in S$, such that $\vec{vw}$.

A digraph is kernel-perfect if every induced subdigraph has a kernel.

Let $f : V(G) \to N$ be a function. A graph G is called f-choosable if a proper coloring can be chosen from any family of lists $\{L(v)\}_{v \in V(G)}$ provided $|L(v)| \geq f(v)$ for every $v \in V(G)$.

Kernel-perfect digraphs and choosability:

Lemma (Bondy-Boppana-Siegel) Let D be a kernel-perfect orientation of G. Then G is f-choosable with $f(v) = 1 + d^+_D(v)$.

Kernel-perfect orientation of $L(K_{n,n})$

Theorem (Galvin, 1995) $\chi'_l(K_{n,n}) = \chi'(K_{n,n})$.

Proof. Give a kernel-perfect orientation to $L(K_{n,n})$ with $\Delta^+ = n - 1$.

$M = W = \{0, 1, 2, \ldots, n-1\}$

$E(K_{n,n}) = V(L(K_{n,n})) = \{ij : i \in M, j \in W\}$

$ij \to i'j$ iff $i + j > i' + j \pmod{n}$

$ij \to ij'$ iff $i + j < i + j' \pmod{n}$

$d^+(ij) = n - 1$ for every $ij \in V(L(K_{n,n}))$

Why do we have a kernel for every $S \subseteq V$?

Detour: Stable Matchings

Bonnie and Clyde is called an unstable pair if

- Bonnie and Clyde are currently not a couple,
- Bonnie prefers Clyde to her current partner, and
- Clyde prefers Bonnie to his current partner.

A perfect matching (of n woman and n man) is a stable matching if it yields no unstable pair.

Theorem. (Gale-Shapley, 1962) There exists a divorce-free society. More precisely: For any preference rankings of n man and n woman there is a stable matching.

Proof. Algorithmic.

The proof of divorce-free society

Proposal Algorithm (Gale-Shapley, 1962)

Input. Preference ranking by each of n man and n woman.

Iteration.
Each man proposes to the woman highest on his list who has not previously rejected him.

IF each woman receives exactly one proposal, THEN
stop and **report** the resulting matching as *stable*.

ELSE

every woman receiving more than one proposal rejects all of them except the one highest on her list.

Every woman receiving at least one proposal says "maybe" to the most attractive proposal she received.

Iterate.

Theorem. The Proposal Algorithm produces a stable matching.

Concluding kernel-perfectness

Why do we have a kernel for every $S \subseteq V$?

Define an appropriate preference list based on S, such that for any stable matching K, $K \cap S$ is a kernel.

Man i prefers woman j to woman j' iff

$ij \in S, ij' \in S$ and $ij \leftarrow ij'$ or

$ij \in S, ij' \notin S$ or

$ij \notin S, ij' \notin S$ and $ij \leftarrow ij'$

Woman j prefers man i to man i' iff

$ij \in S, i'j \in S$ and $ij \leftarrow i'j$ or

$ij \in S, i'j \notin S$ or

$ij \notin S, i'j \notin S$ and $ij \leftarrow i'j$

There goes your kernel

Claim. $K \cap S$ is a kernel for $L(K_{n,n})[S]$

Proof. K is a matching $\Rightarrow K \cap S$ is independent

Suppose there is $ij \in S \setminus K$ which has no outneighbor in $K \cap S$. Let $ij', i'j \in K$.

Then either $ij' \notin S$, or $ij' \in S$ and $ij \leftarrow ij'$. In any case i prefers j to j'.

Similarly either $i'j \notin S$ or $i'j \in S$ and $ij \leftarrow i'j$. In any case j prefers i to i'.

Hence ij is an unstable pair, a contradiction. $\square$

Theorem. (Thomassen) Every planar graph is 5-list colorable.

HW. There is a planar graph which is not 4-list-colorable.

Proof. **Stronger Statement.** Let G be a plane graph with an outer face bounded by cycle C. Suppose that
- two vertices v_1, v_2, $v_1 v_2 \in E(C)$ are colored by two different colors,
- the other vertices of C have 3-element lists assigned to them and
- the internal vertices have 5-element lists assigned to them.

Then the coloring of v_1 and v_2 can be extended properly to the whole G using colors from the assigned lists for each vertex.

Proof. W.l.o.g. every face of G is a triangle, except maybe the outer face.

Induction on $n(G)$. For $n(G) = 3$, $G = K_3$, OK.

For $n(G) > 3$, there are two cases.

Case 1. There is a chord $v_i v_j$ of C.
Cut to two smaller graphs along the chord, color first the piece where both v_1 and v_2 lie, then color the other piece.

Case 2. C has no chord.
Designate two colors $x, y \in L(v_3)$ such that they differ from the color of v_2. Color $G - v_3$ by induction, such that x and y are deleted from the lists of $N(v_3)$. Extend the coloring to v_3.