Jordan Curves

A curve is a subset of $\mathbb{R}^2$ of the form

$$\alpha = \{\gamma(x) : x \in [0,1]\} \ ,$$

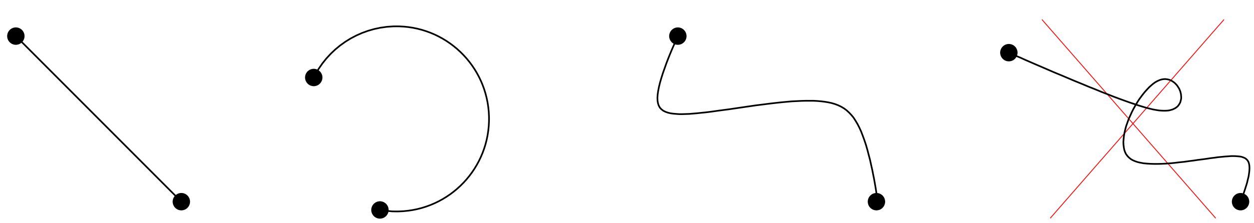
where $\gamma : [0,1] \to \mathbb{R}^2$ is a continuous mapping from the closed interval $[0,1]$ to the plane. $\gamma(0)$ and $\gamma(1)$ are called the *endpoints* of curve α.

A curve is closed if its first and last points are the same. A curve is simple if it has no repeated points except possibly first = last. A closed simple curve is called a Jordan-curve.

Examples: Line segments between $p, q \in \mathbb{R}^2$

$$x \mapsto xp + (1-x)q \ ,$$

circular arcs, Bezier-curves without self-intersection, etc...



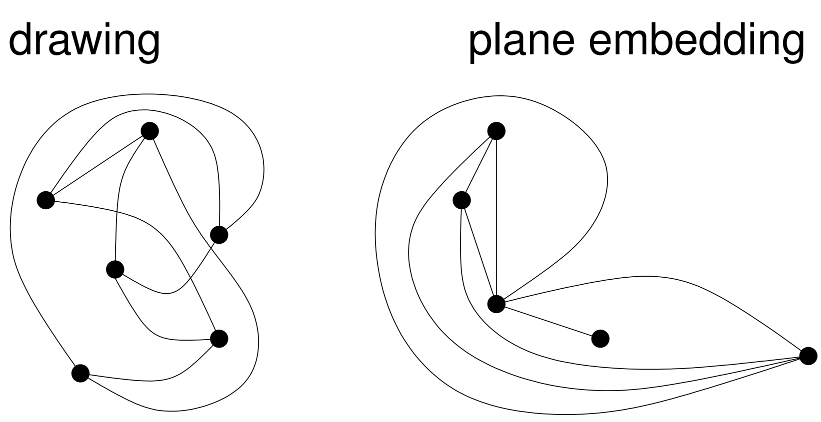
Drawing of graphs

A drawing of a multigraph G is a function f defined on $V(G) \cup E(G)$ that assigns

- a point $f(v) \in I\!R^2$ to each vertex v and
- an $f(u), f(v)$-curve to each edge uv,

such that the images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a crossing.

A multigraph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of G. A planar (multi)graph *together* with a particular planar embedding is called a plane (multi)graph.



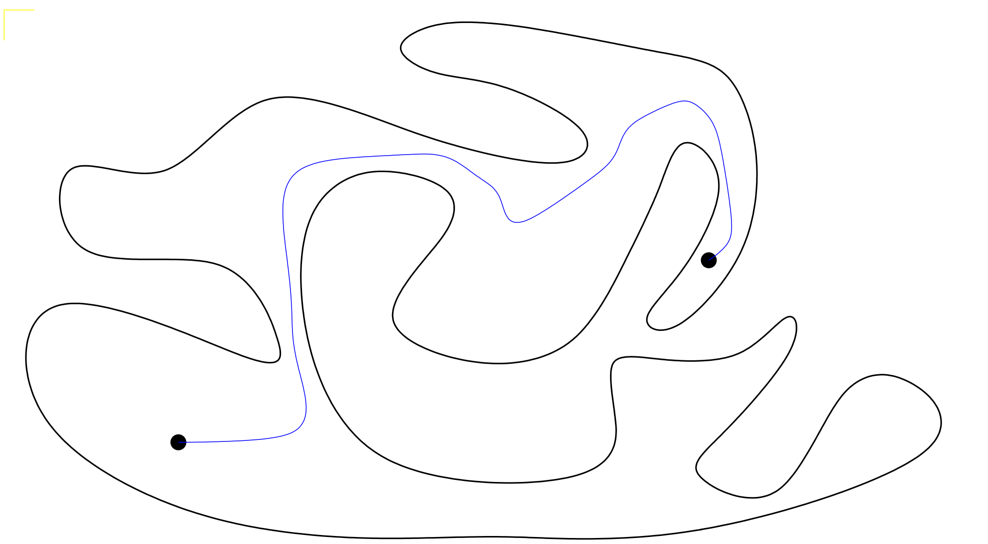
Are there non-planar graphs?

Proposition. K_5 and $K_{3,3}$ cannot be drawn without crossing.

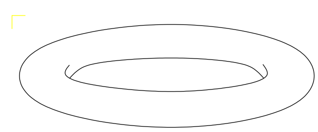
Proof. Define the *conflict graph* of edges.

The unconscious ingredient.

Jordan Curve Theorem. A simple closed curve C partitions the plane into exactly two faces, each having C as boundary.



Not true on the torus!

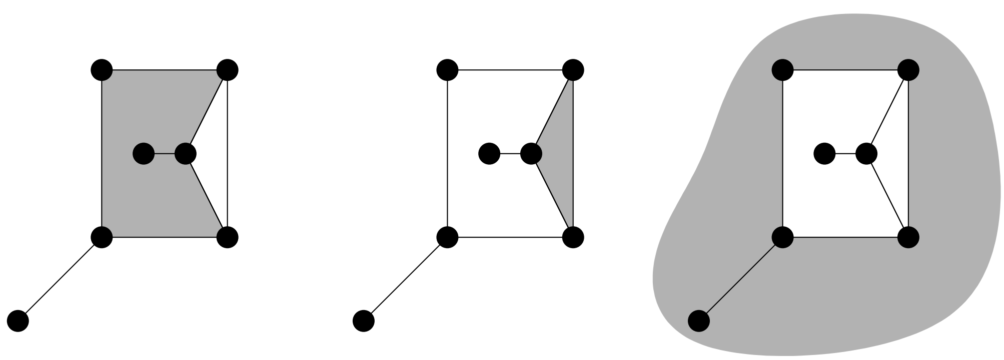


3-a

Regions and faces

An open set in the plane is a set $U \subseteq R^2$ such that for every $p \in U$, all points within some small distance belong to U. A region is an open set U that contains a u, v-curve for every pair $u, v \in U$. The faces of a plane multigraph are the maximal regions of the plane that contain no points used in the embedding.

A finite plane multigraph G has one unbounded face (also called outer face).



Dual graph

Denote the set of faces of a plane multigraph G by $F(G)$ and let $E(G) = \{e_1, \ldots, e_m\}$. Define the dual multigraph G^* of G by

- $V(G^*) := F(G)$
- $E(G^*) := \{e_1^*, \ldots, e_m^*\}$, where the endpoints of e_i^* are the two (not necessarily distinct) faces $f', f'' \in F(G)$ on the two sides of e_i.

Remarks. Multiple edges and/or loops *could* appear in the dual of simple graphs

Different planar embeddings of the *same* planar graph could produce *different* duals.

Proposition. Let $l(F_i)$ denote the length of face F_i in a plane multigraph G. Then

$$2e(G) = \sum l(F_i).$$

Proposition. $e_1, \ldots, e_r \in E(G)$ forms a cycle in G iff $e_1^*, \ldots, e_r^* \in E(G^*)$ forms a minimal nonempty edge-cut in G^*.

Bipartite planar graphs

HW Let G be a plane graph. Then $(G^*)^* \cong G$ iff G connected.

Theorem The following are equivalent for a plane graph G.

- G is bipartite
- Every face of G has even length
- G^* is Eulerian

Outerplanar graphs

A planar graph G is outerplanar if there is an embedding of it in the plane such that all vertices are on the boundary of the outer face.

Proposition. K_4 and $K_{2,3}$ are not outerplanar.

Proposition. If G is simple and outerplanar then $\delta(G) \leq 2$.

HW Every outerplanar graph is 3-colorable

Euler's Formula

Theorem.(Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Proof. Induction on e.

Base Case. If $e = 0$, then $n = k$ and $f = 1$.

Suppose now $e > 0$.

Case 1. G has a cycle.

Delete one edge from a cycle. In the new graph:

$e' = e - 1$, $n' = n$, $f' = f - 1$ (Jordan!), and $k' = k$.

Case 2. G is a forest.

Delete a pendant edge. In the new graph:

$e' = e - 1$, $n' = n$, $f' = f$, and $k' = k + 1$.

Remark. The dual may depend on the embedding of the graph, but the number of faces does *not*.

Application – Platonic solids

- each face is congruent to the same regular convex r-gon, $r \geq 3$
- the same number d of faces meet at each vertex, $d \geq 3$

EXAMPLES: cube, tetrahedron

$$fr = 2e \qquad vd = 2e$$

Substitute into Euler's Formula

$$\frac{2e}{d} - e + \frac{2e}{r} = 2$$

$$\frac{1}{d} + \frac{1}{r} = \frac{1}{2} + \frac{1}{e}$$

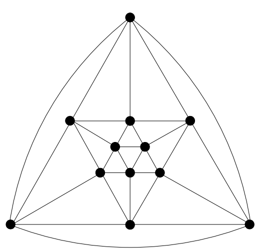
Crucial observation: either d or r is 3.

Possibilities: $r \quad d \quad e \quad f \quad v$

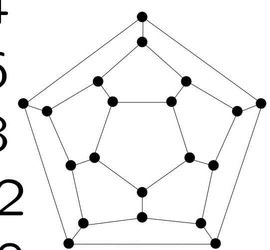
Applications of Euler’s Formula

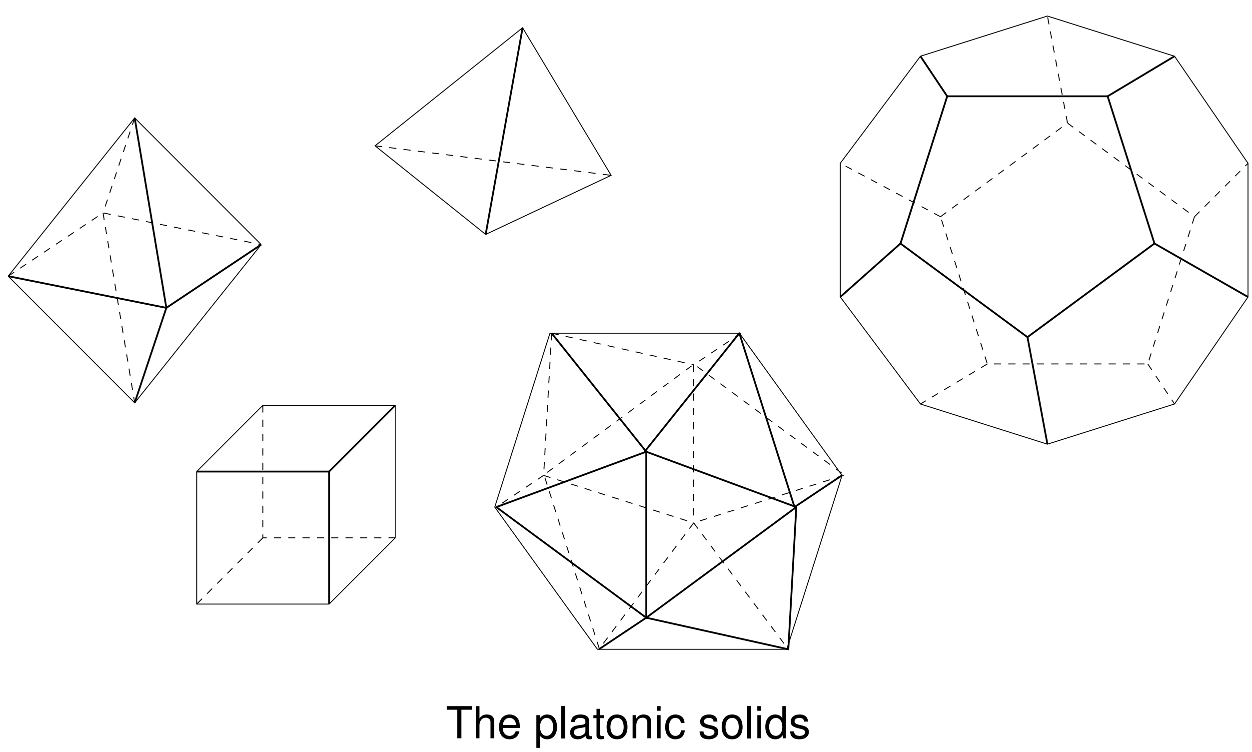
For a convex polytope,

$$\#\text{Vertices} - \#\text{Edges} + \#\text{Faces} = 2$$



Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20





The platonic solids

Number of edges in a planar graphs

Theorem. If G is a simple, planar graph with $n(G) \geq 3$, then $e(G) \leq 3n(G) - 6$.
If also G is triangle-free, then $e(G) \leq 2n(G) - 4$.

Proof. Apply Euler's Formula.

Corollary K_5 and $K_{3,3}$ are non-planar.

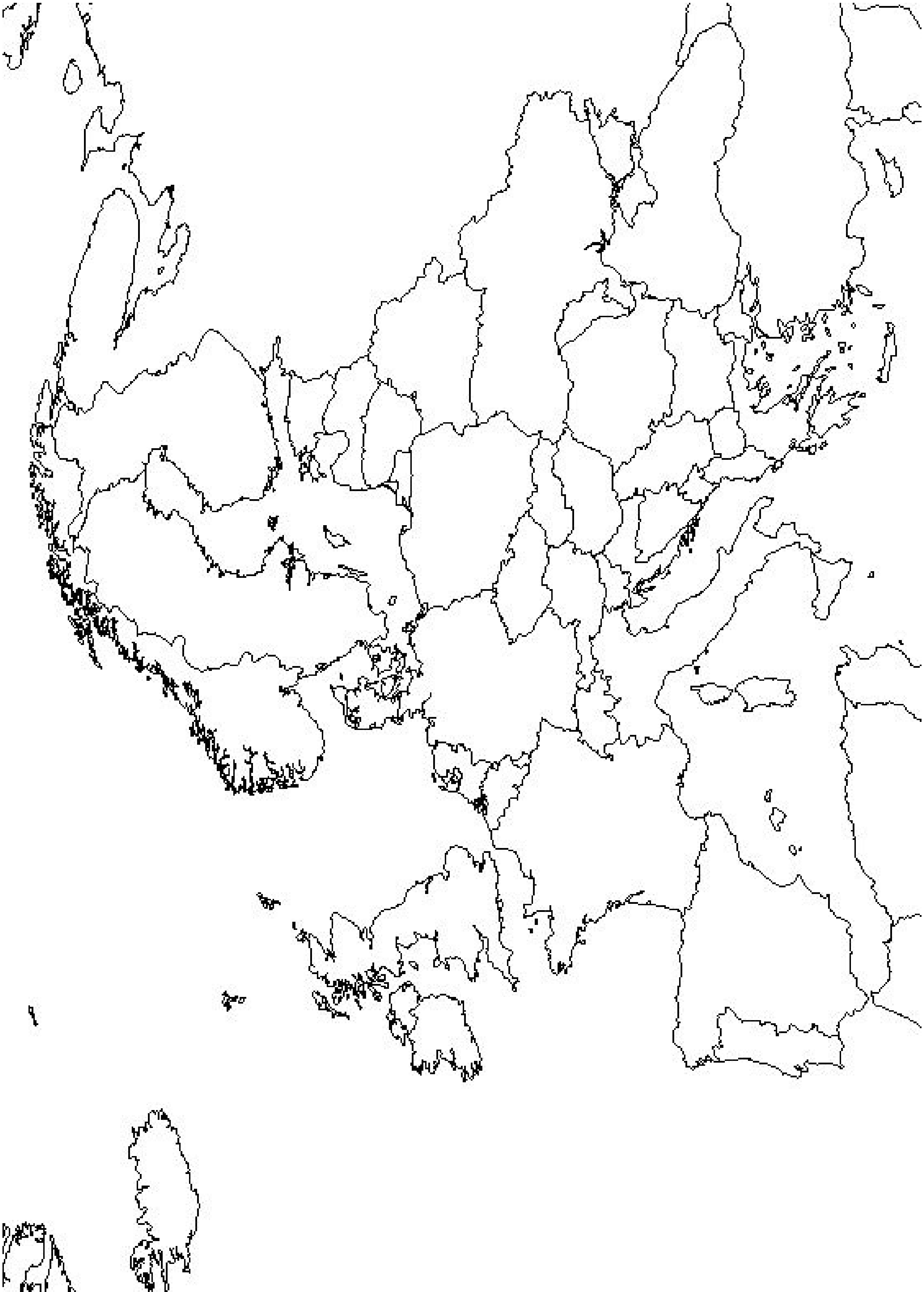
A maximal planar graph is a simple planar graph that is not a spanning subgraph of another planar graph. A triangulation is a simple plane graph where every face is a triangle.

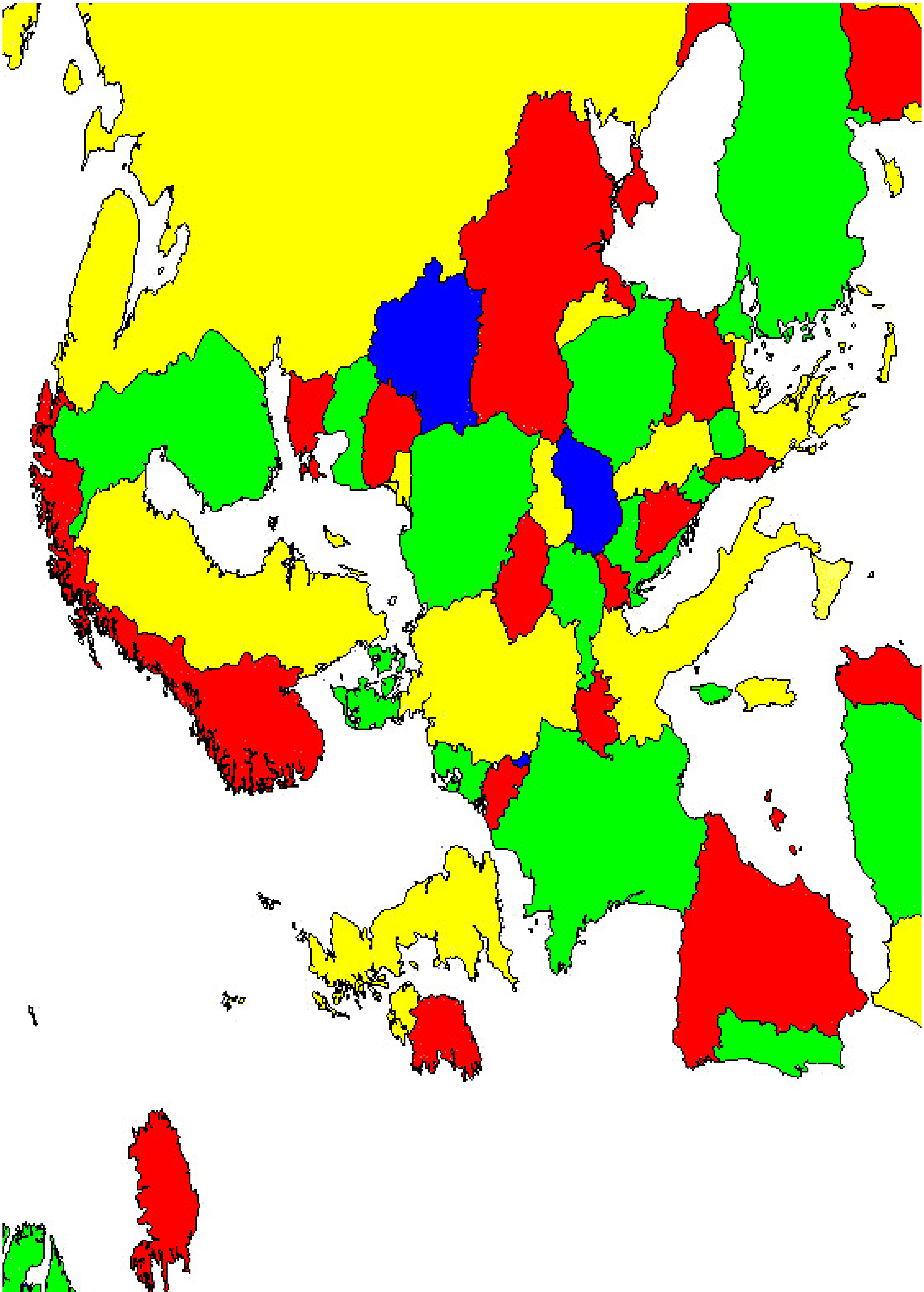
Proposition. For a simple n-vertex plane graph G, the following are equivalent.

$A)$ G has $3n - 6$ edges

$B)$ G is a triangulation.

$C)$ G is a maximal planar graph.





Coloring maps with 5 colors

Five Color Theorem. (Heawood, 1890) If G is planar, then $\chi(G) \leq 5$.

Proof. Take a minimal counterexample.

(i) There is a vertex v of degree at most 5.

(ii) Modify a proper 5-coloring of $G - v$ to obtain a proper 5-coloring of G. A contradiction.

Idea of modification: Kempe chains.

Coloring maps with 4 colors

Four Color Theorem. (Appel-Haken, 1976) For any planar graph G, $\chi(G) \leq 4$.

Idea of the proof.
W.l.o.g. we can assume G is a planar triangulation.
A configuration in a planar triangulation is a separating cycle C (the ring) together with the portion of the graph inside C.
For the Four Color Problem, a set of configurations is an unavoidable set if a minimum counterexample must contain a member of it.
A configuration is reducible if a planar graph containing it cannot be a minimal counterexample.

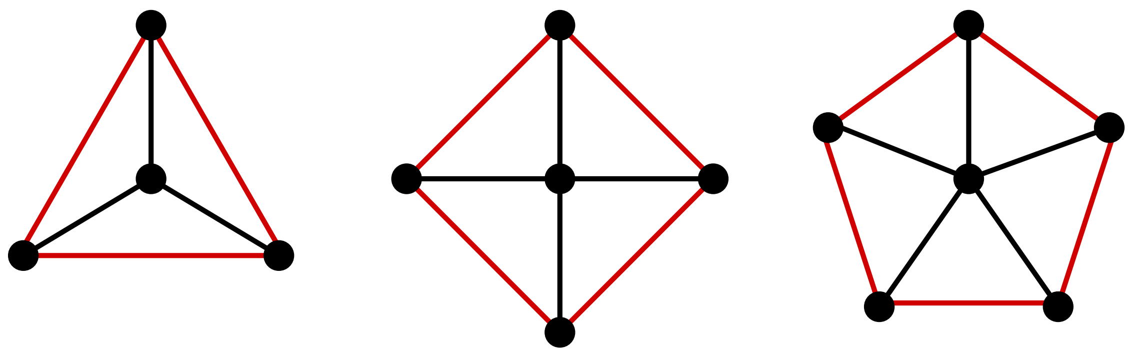
The usual proof attempts to

(i) find a set $\mathcal{C}$ of unavoidable configurations, and

(ii) show that each configuration in $\mathcal{C}$ is reducible.

Proof attempts of the Four Color Theorem

Kempe's original proof tried to show that the unavoidable set



is reducible.

Appel and Haken found an unavoidable set of 1936 of configurations, (all with ring size at most 14) and proved each of them is reducible. (1000 hours of computer time)

Robertson, Sanders, Seymour and Thomas (1996) used an unavoidable set of 633 configuration. They used 32 rules to prove that each of them is reducible. (3 hours computer time)

When is a graph planar?

Theorem(Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Corollary If G is a simple, planar graph with $n(G) \geq 3$, then $e(G) \leq 3n(G) - 6$.
If also G is triangle-free, then $e(G) \leq 2n(G) - 4$.

Corollary K_5 and $K_{3,3}$ are non-planar.

The subdivision of edge $e = xy$ is the replacment of e with a new vertex z and two new edges xz and zy. The graph H' is a subdivision of H, if one can obtain H' from H by a series of edge subdivisions. Vertices of H' with degree at least three are called branch vertices.

Theorem(Kuratowski, 1930) A graph G is planar iff G does not contain a subdivision of K_5 or $K_{3,3}$.

Kuratowski's Theorem

Theorem(Kuratowski, 1930) A graph G is planar iff G does not contain a subdivision of K_5 or $K_{3,3}$.

Proof.
A Kuratowski subgraph of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. A minimal nonplanar graph is a nonplanar graph such that every proper subgraph is planar.

A counterexample to Kuratowski's Theorem constitutes a nonplanar graph that does not contain any Kuratowski subgraph.

Kuratowski's Theorem follows from the following Main Lemma and Theorem.

The spine of the proof

Main Lemma. If G is a graph with fewest edges among counterexamples, then G is 3-connected.

Lemma 1. Every minimal nonplanar graph is 2-connected.

Lemma 2. Let $S = \{x, y\}$ be a separating set of G. If G is a nonplanar graph, then adding the edge xy to some S-lobe of G yields a nonplanar graph.

Main Theorem.(Tutte, 1960) If G is a 3-connected graph with no Kuratowski subgraph, then G has a convex embedding in the plane with no three vertices on a line.

A convex embedding of a graph is a planar embedding in which each face boundary is a convex polygon.

Lemma 3. If G is a 3-connected graph with $n(G) \geq 5$, then there is an edge $e \in E(G)$ such that $G \cdot e$ is 3-connected.

Notation: $G \cdot e$ denotes the graph obtained from G after the contraction of edge e.

Lemma 4. G has no Kuratowski subgraph $\Rightarrow$ $G \cdot e$ has no Kuratowski subgraph.

Proof of Tutte's Theorem

Main Theorem. (Tutte, 1960) If G is a 3-connected graph with no Kuratowski subgraph, then G has a convex embedding in the plane with no three vertices on a line.

Proof. Induction on $n(G)$.
Base case: G is 3-connected, $n(G) = 4 \Rightarrow K_4$.

Let $e \in E$ s.t. $H = G \cdot e$ is 3-connected. (Lemma 3)
Then H has no Kuratowski subgraph. (Lemma 4)
Induction $\Rightarrow$ H has a convex embedding in the plane with no three vertices on a line.

Let $z \in V(H)$ be the contracted e.
$H - z$ is 2-connected $\Rightarrow$ boundary of the face containing z after the deletion of the edges incident to z is a cycle C.

Let $x_1, \ldots, x_k$ be the neighbors of x on C in cyclic order. Note that $|N(x)| \geq 3$ and hence $k \geq 2$.

Denote by $\langle x_i, x_{i+1} \rangle$ the portion of C from x_i to x_{i+1} (including endpoints; indices taken modulo k.)

Let $N_x = N(x) \setminus \{y\}$ and $N_y = N(y) \setminus \{x\}$.

Case 1. $|N_x \cap N_y| \geq 3$.

Let $u, v, w \in N_x \cap N_y$. Then x, y, u, v, w are the branch vertices of a K_5-subdivision in G.

Case 2. $|N_x \cap N_y| \leq 2$.

Since $|N_x \cup N_y| \geq 3$, there is w.l.o.g. a vertex $u \in N_y \setminus N_x$. Let i be such that u is on $\langle x_i, x_{i+1} \rangle$.

Case 2a. N_y is contained in $\langle x_i, x_{i+1} \rangle$.

Then there is an appropriate embedding of G: Placing x in place of z and y sufficiently close to x maintains convexity. (No three vertices are collinear; $|N(x)|, |N(y)| \geq 3$.)

Case 2b. For every i there is a vertex in N_y that is not contained in $\langle x_i, x_{i+1} \rangle$.

Then there must be a $v \in N_y$ that is not on $\langle x_i, x_{i+1} \rangle$ and x, y, x_i, x_{i+1}, u, v are the branch vertices of a $K_{3,3}$-subdivision in G.

Proof of the Lemmas

Lemma 3. G is 3-connected, $n(G) \geq 5 \Rightarrow$ there is an edge $e \in E(G)$ such that $G \cdot e$ is 3-connected.

Proof. Suppose G is 3-connected and for every $e \in E$, $G \cdot e$ is NOT 3-connected.

For edge $e = xy$, the vertex z is the mate of xy if $\{x, y, z\}$ is a cut in G.

Choose $e = xy$ and their mate z such that $G - \{x, y, z\}$ has a component H, whose order is as large as possible.

Let H' be another component of $G - \{x, y, z\}$ and let $u \in V(H')$ be the neighbor of z (There IS one!). Let v be the mate of uz.

$V(H) \cup \{x, y\} \setminus \{v\}$ is connected in $G - \{z, u, v\}$ contradicting the maximality of H. $\square$

Lemma 4. G has no Kuratowski subgraph $\Rightarrow$ $G \cdot e$ has no Kuratowski subgraph.

Proof. Suppose $G \cdot e$ contains a Kuratowski subgraph H. Then

- $z \in V(H)$
- z is a branchvertex of H
- $|N_H(z)| = 4$ and $|N_H(z) \cap N_G(x)|, |N_H(z) \cap N_G(y)| \geq 2$

Then H is the subdivision of K_5 $\Rightarrow$ G contains a subdivision of $K_{3,3}$, a contradiction. $\square$

Minors

K_7 is a toroidal graph (it is embeddable on the torus), K_8 is not. What else is not? For the torus there is NO equivalent version of Kuratowski's characterization with a finite number of forbidden subdivisions. Any such characterization must lead to an infinite list.

A weaker concept: Minors.

Graph G is called a minor of graph H is if G can be obtained from H by a series of edge and vertex deletions and edge contractions. Graph H is also called a G-minor

Example: K_5 is a minor of the Petersen graph P, but P does not contain a K_5-subdivision.

The Graph Minor Theorem

Theorem. (Robertson and Seymour, 1985-2005) In any infinite list of graphs, some graph is a minor of another.

Proof: more than 500 pages in 20 papers.

Corollary For any graph property that is closed under taking minors, there exists finitely many minimal forbidden minors.

Homework. Wagner's Theorem. Every nonplanar graph contains either a K_5 or $K_{3,3}$-minor.

For embeddability on the projective plane, it is known that there are 35 minimal forbidden minors. For embeddability on the torus, we don't know the exact number of minimal forbidden minors; there are more than 800 known.