How to find a $T_{rt,r}$ These notes augment the transparencies for the second proof of the Erdős-Stone Theorem (via the Regularity Lemma). The notation is the one used there.

Our main tool in finding a $T_{rt,t}$ in $G[V_1 \cup \cdots \cup V_r]$ is the Degree Lemma.

Degree Lemma *Let (A, B) be an ε-regular pair with $d(A, B) \geq d$ Let $Y \subseteq B$ be a subset with $|Y| \geq \varepsilon|B|$. Then for the number of vertices of A with small degree into Y we have*

$$|\{v \in A : d_Y(v) < (d - \varepsilon)|Y|\}| < \varepsilon|A|.$$

Proof. Otherwise the subsets $Y \subseteq B$ and $\{v \in A : d_Y(v) < (d-\varepsilon)|Y|\} \subseteq A$ would contradict the ε-regularity of (A, B). □

We start by finding t vertices in V_1, which have a "large" common neigborhood in each of the other sets $V_2, \ldots, V_r$. (Here and later "large" always will mean size that is a positive fraction of the size $\tilde{n}$ of the V_i. This positive fraction will be extremely small compared to the parameter γ of the theorem, it will be a tiny-tiny positive number depending on γ, but not depending on n.)

We find these t vertices $v_{1,1}, v_{1,2}, \ldots, v_{1,t} \in V_1$ one by one. For the first vertex $v_{1,1}$ we just want to make sure that it has a relatively large neighborhood in each of the sets $V_2, \ldots, V_r$. How large neighborhoods can we hope for? Well, the density of edges in the whole graph is at least d (remember: after using the Regularity Lemma we kept only those edges of G that went between ϵ-regular pairs of density at least d), that's more or less an upper limit to our hopes. We will show that we can get very close to this: as a consequence of ϵ-regularity we will be able to find a first vertex $v_{1,1} \in V_1$ which has at least $(d-\epsilon)\tilde{n}$ neighbors in each of the sets $V_2, \ldots, V_r$. This is possible, because according to the Degree Lemma there are at most $\epsilon|V_1| = \epsilon\tilde{n}$ vertices in V_1 with small degree into V_2, $\epsilon\tilde{n}$ vertices with small degree into V_3, etc, ... So, all together there are **at least** $\tilde{n} - (r-1)\epsilon\tilde{n}$ vertices in V_1 that have degree **at least** $(d - \epsilon)\tilde{n}$ in each of $V_2, \ldots, V_r$. Hence if ϵ is chosen to be strictly less than $\frac{1}{r-1}$, then **there is** such a vertex in V_1; we choose an arbitrary one to be $v_{1,1}$. So we will make sure at the end that $\epsilon < \frac{1}{r-1}$.

We need now a $v_{1,2} \in V_1$ which has a large common neighborhood with (our already chosen) $v_{1,1}$ into each of the sets $V_2, \ldots, V_r$. To find such a vertex we use the Degree Lemma for $A = V_1$ and $Y = \Gamma_{V_2}(\{v_{1,1}\})$, then for $A = V_1$ and $Y = \Gamma_{V_3}(\{v_{1,1}\})$, and so on, ..., for $A = V_1$ and $Y = \Gamma_{V_r}(\{v_{1,1}\})$. To apply the Lemma we need that $|\Gamma_{V_i}(\{v_{1,1}\})| > \epsilon\tilde{n}$. This can be assured by choosing an appropriately small ϵ: we know already, because of the selection of $v_{1,1}$, that $|\Gamma_{V_i}(\{v_{1,1}\})| > (d - \epsilon)\tilde{n}$ for each $i = 2, \ldots, r$. So all we need is that ϵ is chosen such that $d-\epsilon > \epsilon$. Than the Degree Lemma tells us that for every $i = 2, 3 \ldots, r$, there are at most $\epsilon|V_1|$ vertices in V_1 whose degree into $\Gamma_{V_i}(\{v_{1,1}\})$ is smaller than $(d-\epsilon)|\Gamma_{V_i}(\{v_{1,1}\})| \geq (d-\epsilon)^2\tilde{n}$. So there are **at least** $\tilde{n} - (r-1)\epsilon\tilde{n}$ vertices in V_1 whose degree into each $\Gamma_{V_i}(\{v_{1,1}\})$ is **at least** $(d-\epsilon)^2\tilde{n}$. If $\tilde{n} - (r-1)\epsilon\tilde{n} > 1$, then we can select one that is different from $v_{1,1}$. This will be $v_{1,2}$ and we have $|\Gamma_{V_i}(\{v_{1,1}, v_{1,2}\})| > (d-\epsilon)^2\tilde{n}$ for each $i = 2, \ldots, r$.

We select $v_{1,3}, v_{1,4}, \ldots, v_{1,t}$ similarly; for $v_{1,j}$ we want a vertex whose neighborhood into the common neighborhoods $\Gamma_{V_i}(\{v_{1,1}, \ldots, v_{1,j-1}\})$ of the already selected vertices is large for each $i = 2, \ldots, r$. Here by "large" we mean $(d-\epsilon)^j\tilde{n}$ and we assume by indcution that the common neighborhood of the vertices $v_{1,1}, \ldots, v_{1,j-1}$ into each of the V_i, $i = 2, \ldots, r$, contains at least $(d-\epsilon)^{j-1}\tilde{n}$ vertices. We use the Degree Lemma for $A = V_1$ and $Y = \Gamma_{V_2}(\{v_{1,1}, \ldots, v_{1,j-1}\})$, then for $A = V_1$ and $Y = \Gamma_{V_3}(\{v_{1,1} \ldots, v_{1,j-1}\})$, and so on, $\ldots$, for $A = V_1$ and $Y = \Gamma_{V_r}(\{v_{1,1}, \ldots, v_{1,j-1}\})$. To apply the Lemma we need that $|\Gamma_{V_i}(\{v_{1,1}, \ldots, v_{1,j-1}\})| > \epsilon\tilde{n}$. This can be assured by choosing an appropriately small ϵ: we know already by induction, because of the selection of $v_{1,1}, \ldots, v_{1,j-1}$, that $|\Gamma_{V_i}(\{v_{1,1}, \ldots, v_{1,j-1}\})| > (d-\epsilon)^{j-1}\tilde{n}$ for each $i = 2, \ldots, r$. So all we need is that ϵ is chosen such that $(d-\epsilon)^{j-1} > \epsilon$. Than the Degree Lemma tells us that for every $i = 2, 3, \ldots, r$, there are at most $\epsilon|V_1|$ vertices whose degree into $\Gamma_{V_i}(\{v_{1,1}, \ldots, v_{1,j-1}\})$ is smaller than $(d-\epsilon)|\Gamma_{V_i}(\{v_{1,1}, \ldots, v_{1,j-1}\})| \geq (d-\epsilon)^j\tilde{n}$. So there are **at least** $\tilde{n} - (r-1)\epsilon\tilde{n}$ vertices in V_1 whose degree into each $\Gamma_{V_i}(\{v_{1,1}, \ldots, v_{1,j-1}\})$ is **at least** $(d-\epsilon)^j\tilde{n}$. If $\tilde{n} - (r-1)\epsilon\tilde{n} > j-1$, then we can select one that is different from each of $v_{1,1}, \ldots, v_{1,j-1}$; this will be the vertex $v_{1,j}$.

In conclusion, we successfully end this process of selecting a t-element set $S_1 = \{v_{1,1}, \ldots, v_{1,t}\} \subseteq V_1$ with a common neighborhood $\Gamma_{V_i}(S_1)$ of size at least $(d-\epsilon)^t\tilde{n}$ for each $i = 2, \ldots, r$, provided the following two conditions hold for ϵ:

(1) $(d-\epsilon)^{t-1}\tilde{n} > \epsilon\tilde{n}$
(this implies $(d-\epsilon)^{j-1}\tilde{n} > \epsilon\tilde{n}$ for each $j = 1, \ldots, t$ and hence guarantees that the Degree Lemma can be applied in each step).

(2) $\tilde{n} - (r-1)\epsilon\tilde{n} > t-1$
(this implies that $\tilde{n} - (r-1)\epsilon\tilde{n} > j-1$ for each $j = 1, \ldots, t$, and hence makes sure that there is at least one vertex $v_{1,j} \in V_1$ which is different from the previously selected $v_{1,1}, \ldots, v_{1,j-1}$ and is **not** part of the at most $\epsilon\tilde{n}$ vertices which have small degree into $\Gamma_{V_i}(v_{1,1}, \ldots, v_{1,j-1})$ for some $r = 2, \ldots r$.)

We conclude that provided the above two conditions hold, we can find a subset $S_1 \subseteq V_1$ of size t such that its common neighborhoods $\Gamma_{V_i}(S_1)$ into V_i has size at least $(d-\epsilon)^t\tilde{n}$ for each $i = 2, \dots r$.

Now we go on analogously and find a t-subset $S_2 \subseteq \Gamma_{V_2}(S_1)$ with large common neighborhoods into $\Gamma_{V_i}(S_1)$. This is done completely analogously by replacing in our argument the index set $\{1, 2, 3, \dots, r\}$ with $\{2, 3, \dots, r\}$, the V_i with $\Gamma_{V_i}(S_1)$. The conditions change as follows:

- $(d-\epsilon)^{2t-1}\tilde{n} > \epsilon\tilde{n}$ and
- $(d-\epsilon)^t\tilde{n} - (r-2)\epsilon\tilde{n} > t-1$

The first condition will guarantee that the common neighborhood of the vertices $v_{1,1}, \dots, v_{1,t}, v_{2,1}, \dots, v_{2,j-1}$ into V_i is large enough so the Degree Lemma can be applied.

The second condition ensures that inside $\Gamma_{V_2}(S_1)$ we can select a vertex $v_{2,j}$ which is **not** one of the previously selected vertices $v_{2,1}, \dots, v_{2,j-1}$ and **not** among the $\epsilon\tilde{n}$ vertices which have small degree into $\Gamma_{V_i}(S_1 \cup \{v_{2,1}, \dots, v_{2,j-1}\})$ for some $i = 3, \dots, r$.

We proceed analogously to find $S_i \subseteq \Gamma_{V_i}(S_1 \cup \dots \cup S_{i-1})$ with large common neighborhoods into each $\Gamma_{V_j}(S_1 \cup \dots \cup S_{i-1})$, $j > i$. Quantitatively, we mean that $|\Gamma_{V_i}(S_1 \cup \dots \cup S_i)| > (d-\epsilon)^{it}\tilde{n}$ for every $j = i+1, \dots, r$. The conditions for this look like as follows:

(1) $(d-\epsilon)^{it-1}\tilde{n} > \epsilon\tilde{n}$ and

(2) $(d-\epsilon)^{(i-1)t}\tilde{n} - (r-i)\epsilon\tilde{n} > t-1$

In the last step (for $j = r-1$), we found a t-subset $S_{r-1} \subseteq \Gamma_{V_{r-1}}(S_1 \cup \dots \cup S_{r-2})$ with a large common neighborhood into $\Gamma_{V_r}(S_1 \cup \dots \cup S_{r-2})$, i.e., $|\Gamma_{V_r}(S_1 \cup \dots \cup S_{r-1})| > (d-\epsilon)^{(r-1)t}$.

Now if

$$(d-\epsilon)^{(r-1)t} > t-1$$

then we can find a t-element set $S_r \subseteq \Gamma_{V_r}(S_1 \cup \dots \cup S_{r-1})$, which completes the construction of the $T_{rt,r}$ we are after ($G[S_1 \cup \dots \cup S_r]$ is isomorphic to $T_{rt,r}$).

Let us see now which conditions should ϵ satisfy. The strongest of the conditions of type (1) is for $i = r-1$ and it reads $(d-\epsilon)^{(r-1)t-1}\tilde{n} > \epsilon\tilde{n}$.

To guarantee all conditions of type (2) we will take

$$(d-\epsilon)^{(r-1)t}\tilde{n} - (r-1)\epsilon\tilde{n} > t-1.$$

This is certainly much stronger than the ones we got from conditions of type (1), so at the end that's the only condition about ε we will worry about.

To satisfy the condition, first of all we take an ε, so the left hand side is positive, for example take an ε, so $(d-\epsilon)^{(r-1)t} > r\varepsilon$, which makes the left hand side at least $\varepsilon\tilde{n}$. Then we ensure, by choosing a large enough $\tilde{n}$, that $\epsilon\tilde{n} > t-1$.

For the density d we choose for example

$$d = \frac{\gamma}{6}$$

For the lower bound m on the number of parts the Regularity Lemma creates we choose

$$m = \frac{6}{\gamma}.$$

This m is also large enough for using Turán's Theorem earlier when we found our K_r in the regularity graph $R(\mathcal{P}, d)$ of G:

$$ex(K_m, r) \leq \left(1 - \frac{1}{r-1}\right)\binom{m}{2} + \frac{m}{2} < \left(1 - \frac{1}{r-1}\right)\binom{m}{2} + \frac{\gamma}{2}m^2.$$

For ϵ we take for example

$$\varepsilon = \frac{1}{r}\left(\frac{d}{2}\right)^{t(r-1)}.$$

Then the condition

$$d + 2\varepsilon + \frac{1}{m} < \frac{\gamma}{2},$$

which guarantees that most of the edges of G go in between ϵ-regular pairs of density d, is satisfied, and we also have

$$(d-\varepsilon)^{(r-1)t} > \left(d - \frac{d}{2}\right)^{(r-1)t} = r\varepsilon.$$

which is need for the above.

Finally we choose

$$N(r,t,\gamma) = \frac{t-1}{\varepsilon(1-\varepsilon)}M(\varepsilon, m),$$

so for every $n > N(r,t,\gamma)$ we have that

$$\varepsilon\tilde{n} \geq \frac{1-\varepsilon}{k}n \geq \frac{1-\varepsilon}{M(\varepsilon,m)}n > \frac{1-\varepsilon}{M(\varepsilon,m)}N(r,t,\gamma) = t-1.$$

The actual order of the proof: The theorem gave us parameters r, t and γ. From this first we defined d, m and ε. For m and ε we used the Regularity Lemma, and that gave us back $M(\varepsilon, m)$. Using this we chose our $N(r,t,\gamma)$. Then the proof started: we are given a graph G, we apply the Regularity Lemma, we define the regularity graph, we find in it the K_r and finally in the subgraph $G[V_1 \cup \cdots \cup V_r]$ we find $T_{rt,r}$.