# RECAP — Matchings

A matching is a set of (non-loop) edges with no shared endpoints. The vertices incident to an edge of a matching $M$ are saturated by $M$, the others are unsaturated. A perfect matching of $G$ is matching which saturates all the vertices.

*Examples.* $K_{n,m}$, $K_n$, Petersen graph, $Q_k$; graphs without perfect matching

The size of the largest matching in $G$ is denoted by $\alpha'(G)$.

A vertex cover of $G$ is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge.
The size of the smallest vertex cover in $G$ is $\beta(G)$.

**Claim.** For every graph $G$, $\beta(G) \geq \alpha'(G)$.

# Certificates

Suppose we knew that in some graph $G$ with $1121$ edges on $200$ vertices, a particular set of $87$ edges is (one of) the largest matching one could find. How could we convince somebody about this?

Once the particluar $87$ edges are shown, it is easy to check that they are a matching, indeed.

But why isn't there a matching of size $88$? Verifying that none of the $\binom{1121}{88}$ edgesets of size $88$ forms a matching could take some time...

If we happen to be so lucky, that we are able to exhibit a vertex cover of size $87$, we are saved. It is then reasonable to check that all $1121$ edges are covered by the particular set of $87$ vertices.

Exhibiting a vertex cover of a certain size **proves** that no larger matching can be found.

## Certificate for bipartite graphs — Take 1

**1. Correctness** of the certificate:

A vertex cover $Q \subseteq V(G)$ is a certificate proving that no matching of $G$ has size larger than $|Q|$.
That is: $\beta(G) \geq \alpha'(G)$, valid for every graph.

**2. Existence** of optimal certificate for bipartite graphs:

**Theorem.** (König (1931), Egerváry (1931))
If $G$ is bipartite then $\beta(G) = \alpha'(G)$.

König's Theorem $\Rightarrow$ For bipartite graphs there always exists a vertex cover proving that a particular matching of maximum size is really maximum.

**Remark.** This is NOT the case for general graphs: $C_5$.

## Certificate for bipartite graphs — Take 2

Let $G$ be a bipartite graph with partite sets $X$ and $Y$.

**1. Correctness** of the certificate:

A subset $S \subseteq X$ is a certificate proving that the largest matching in $G$ has size at most $|X| - |S| + |N(S)|$.

**2. Existence** of optimal certificate:

**Theorem** (Marriage Theorem; Hall, 1935) There is a matching in $G$ saturating $X$ iff $|N(S)| \geq |S|$ for every $S \subseteq X$.

**Corollary** $\alpha'(G) = |X|$ or there exists a subset $S \subseteq X$, such that $\alpha'(G) = |X| - |S| + |N(S)|$.

*Proof.* Homework.

**Problem:** Certificate makes sense for bipartite graphs only.
**Goal:** Find a certificate for general graphs.

## Matchings in general graphs

An odd component is a connected component with an odd number of vertices. Denote by $o(G)$ the number of odd components of a graph $G$.

**Theorem.** (Tutte, 1947) A graph $G$ has a perfect matching iff $o(G - S) \leq |S|$ for every subset $S \subseteq V(G)$.

*Proof.*

$\Rightarrow$ Easy.

$\Leftarrow$ (Lovász, 1975) Consider a counterexample $G$ with the maximum number of edges.

*Claim.* $G + xy$ has a perfect matching for any $xy \notin E(G)$.

## Proof of Tutte's Theorem — Continued

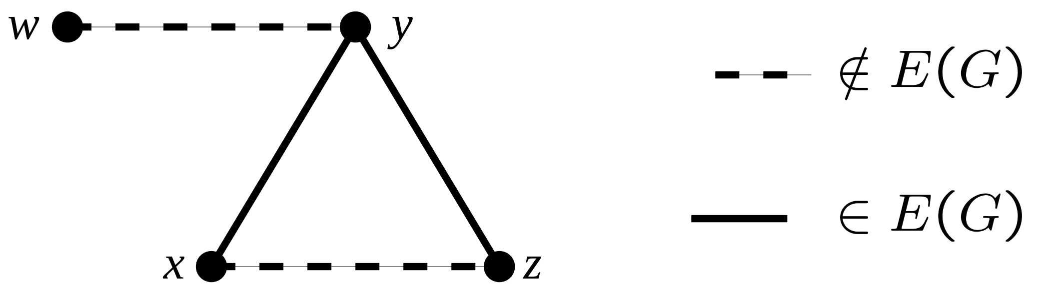
Define $U := \{v \in V(G) : d_G(v) = n(G) - 1\}$

*Case 1.* $G - U$ consists of disjoint cliques.

*Proof:* Straightforward to construct a perfect matching of $G$.

*Case 2.* $G - U$ is not the disjoint union of cliques.

*Proof:* Derive the existence of the following subgraph.




Obtain contradiction by constructing a perfect matching $M$ of $G$ using perfect matchings $M_1$ and $M_2$ of $G+xz$ and $G + yw$, respectively.

## Corollaries

**Corollary.** (Berge,1958) For a subset $S \subseteq V(G)$ let $d(S) = o(G - S) - |S|$. Then

$$2\alpha'(G) = \min\{n - d(S) : S \subseteq V(G)\}.$$

*Proof.* ($\leq$) Easy.
($\geq$) Apply Tutte's Theorem to $G \vee K_d$.

**Corollary.** (Petersen, 1891) Every 3-regular graph with no cut-edge has a perfect matching.

*Proof.* Check Tutte's condition. Let $S \subseteq V(G)$.
Double-count the number of edges between an $S$ and the odd components of $G - S$.
Observe that between any odd component and $S$ there are at least three edges.