Recall: Connectivity

A separating set (or vertex cut) of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. For $G \neq K_n$, the connectivity of G is $\kappa(G) := min\{|S| : S \text{ is a vertex cut}\}$. By definition, $\kappa(K_n) := n - 1$. A graph G is k-connected if there is no vertex cut of size $k - 1$. (i.e. $\kappa(G) \geq k$)

Examples.

$$\kappa(K_{n,m}) = \min\{n, m\}$$
$$\kappa(Q_d) = d$$

Recall: Edge-connectivity

An edge cut of a multigraph G is an edge-set of the form $[S, \bar{S}]$, with $\emptyset \neq S \neq V(G)$ and $\bar{S} = V(G) \setminus S$.

For $S, T \subseteq V(G)$, $[S, T] := \{xy \in E(G) : x \in S, y \in T\}$.

The edge-connectivity of G is

$$\kappa'(G) := \min\{ |[S, \bar{S}]| : [S, \bar{S}] \text{ is an edge cut}\}.$$

A graph G is k-edge-connected if there is no edge cut of size $k - 1$ (i.e. $\kappa'(G) \geq k$).

Theorem. (Whitney, 1932) If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Homework. Example of a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = m$, for any $0 < k \leq l \leq m$.

HW G is 3-regular $\Rightarrow \kappa(G) = \kappa'(G)$.

Characterization of 2-connected graphs

Theorem. (Whitney,1932) Let G be a graph, $n(G) \geq 3$. Then G is 2-connected iff for every $u, v \in V(G)$ there exist two internally disjoint u, v-paths in G.

Theorem. Let G be a graph with $n(G) \geq 3$. Then the following four statements are equivalent.

(i) G is 2-connected

(ii) For all $x, y \in V(G)$, there are two internally disjoint x, y-path.

(iii) For all $x, y \in V(G)$, there is a cycle through x and y.

(iv) $\delta(G) \geq 1$, and every pair of edges of G lies on a common cycle.

Expansion Lemma. Let G' be a supergraph of a k-connected graph G obtained by adding one vertex to $V(G)$ with at least k neighbors.
Then G' is k-connected as well.

Menger's Theorem

Given $x, y \in V(G)$, a set $S \subseteq V(G) \setminus \{x, y\}$ is an x, y-separator (or an x, y-cut) if $G - S$ has no x, y-path.

A set $\mathcal{P}$ of paths is called pairwise internally disjoint (p.i.d.) if for any two path $P_1, P_2 \in \mathcal{P}$, P_1 and P_2 have no common internal vertices.

Define

$$\kappa(x, y) := \min\{|S| : S \text{ is an } x, y\text{-cut,}\} \text{ and}$$
$$\lambda(x, y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.i.d. } x, y\text{-paths}\}$$

Local Vertex-Menger Theorem (Menger, 1927) Let $x, y \in V(G)$, such that $xy \notin E(G)$. Then

$$\kappa(x, y) = \lambda(x, y).$$

Corollary (Global Vertex-Menger Theorem) A graph G is k-connected iff for any two vertices $x, y \in V(G)$ there exist k p.i.d. x, y-paths.

Proof: **Lemma.** For every $e \in E(G)$, $\kappa(G - e) \geq \kappa(G) - 1$.

Edge-Menger

Given $x, y \in V(G)$, a set $F \subseteq E(G)$ is an x, y-disconnecting set if $G - F$ has no x, y-path. Define

$$\kappa'(x,y) := \min\{|F| : F \text{ is an } x,y\text{-disconnecting set,}\}$$
$$\lambda'(x,y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.e.d.}^* \ x,y\text{-paths}\}$$

* p.e.d. means pairwise edge-disjoint

Local Edge-Menger Theorem For all $x, y \in V(G)$,

$$\kappa'(x,y) = \lambda'(x,y).$$

Proof. Apply Menger's Theorem for the line graph of G', where $V(G') = V(G) \cup \{s,t\}$ and $E(G') = E(G) \cup \{sx, yt\}$.

The line graph $L(G)$ of a graph G is defined by
$V(L(G)) := E(G)$,
$E(L(G)) := \{ef : e \text{ and } f \text{ share an endpoint}\}$.

Corollary (Global Edge-Menger Theorem) Multigraph G is k-edge-connected iff there is a set of k p.e.d. x, y-paths for any two vertices x and y.

Network flows

Network (D, s, t, c); D is a directed multigraph, $s \in V(D)$ is the source, $t \in V(D)$ is the sink, $c : E(D) \to \mathbb{R}^{+} \cup \{0\}$ is the capacity.

Flow f is a function, $f : E(D) \to \mathbb{R}$

$$f^{+}(v) := \sum_{v \to u} f(vu)$$

$$f^{-}(v) := \sum_{u \to v} f(uv).$$

Flow f is feasible if

(i) $f^{+}(v) = f^{-}(v)$ for every $v \neq s, t$ (conservation constraints), and

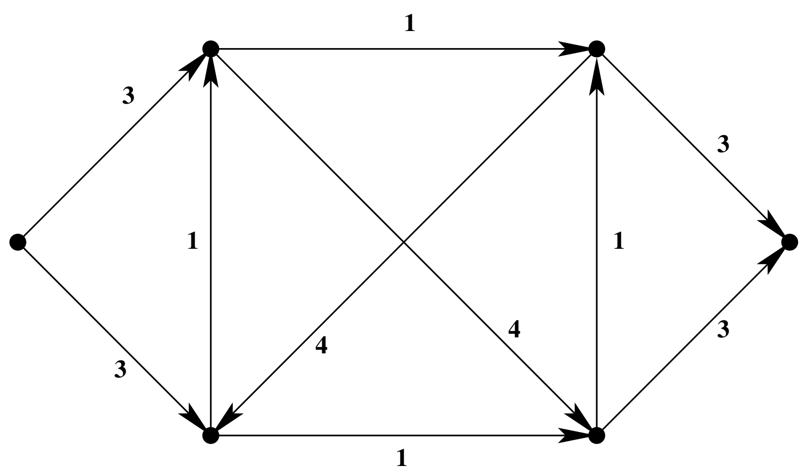
(ii) $0 \leq f(e) \leq c(e)$ for every $e \in E(D)$ (capacity constraints).

value of flow, $val(f) := f^{-}(t) - f^{+}(t)$.

maximum flow: feasible flow with maximum value

Example

0-flow



f-augmenting path

G: underlying undirected graph of network D

s, t-path P in G is an f-augmenting path, if $s = v_0, e_1, v_1, e_2 \ldots v_{k-1}, e_k, v_k = t$ and for every e_i

(i) $f(e_i) < c(e_i)$ provided e_i is "forward edge"

(ii) $f(e_i) > 0$ provided e_i is "backward edge"

Tolerance of P is $\min\{\epsilon(e) : e \in E(P)\}$, where
$\epsilon(e) = c(e) - f(e)$ if e is forward, and
$\epsilon(e) = f(e)$ if e is backward.

Lemma. Let f be feasible and P be an f-augmenting path with tolerance z. Define
$f'(e) := f(e) + z$ if e is forward,
$f'(e) := f(e) - z$ if e is backward.
$f'(e) := f(e)$ if $e \notin E(P)$,
Then f' is feasible with $val(f') = val(f) + z$.

Characterization of maximum flows

Characterization Lemma. Feasible flow f is of maximum value iff there is NO f-augmenting path.

Proof. $\Rightarrow$ Easy.
$\Leftarrow$ Suppose f has no augmenting path.

$S := \{v \in V(D) : \exists\, f$-augmenting path* from s to $v\}$.

Then $t \notin S$ and

$$\sum_{e\in[S,\bar{S}]} c(e) = \sum_{e\in[S,\bar{S}]} f(e) - \sum_{e\in[\bar{S},S]} f(e).$$

We feel, that

(1) $val(f^*) \leq \sum_{e\in[S,\bar{S}]} c(e)$ for any feasible flow f^*, and

(2) $val(f) = \sum_{e\in[Q,\bar{Q}]} f(e) - \sum_{e\in[\bar{Q},Q]} f(e)$, for any $Q \subseteq V(D), s \in Q, t \notin Q$.

Right? Let's see

The value of feasible flow ______ Proof of (2)

Lemma If f is any feasible flow, $s \in Q$, $t \notin Q$, then

$$\sum_{e \in [Q,\bar{Q}]} f(e) - \sum_{e \in [\bar{Q},Q]} f(e) = val(f).$$

Proof. By induction on $|\bar{Q}|$. If $|\bar{Q}| = 1$ then $\bar{Q} = \{t\}$ and by definition $f^-(t) - f^+(t) = val(f)$.

Let $|\bar{Q}| \geq 2$ and let $x \in \bar{Q}$, $x \neq t$.
Define $R = Q \cup \{x\}$. Since $|\bar{R}| < |\bar{Q}|$, by induction

$$\begin{aligned} val(f) &= \sum_{e \in [R,\bar{R}]} f(e) - \sum_{e \in [\bar{R},R]} f(e) \\ &= \sum_{e \in [Q,\bar{Q}]} f(e) - \sum_{e \in [\bar{Q},Q]} f(e) + \sum_{u \in Q} f(xu) \\ &\quad - \sum_{u \in Q} f(ux) + \sum_{v \in \bar{R}} f(xv) - \sum_{v \in \bar{R}} f(vx) \\ &= \sum_{e \in [Q,\bar{Q}]} f(e) - \sum_{e \in [\bar{Q},Q]} f(e) + f^+(x) - f^-(x) \end{aligned}$$

Remark. $val(f) = f^+(s) - f^-(s)$.

Source/sink cut $[S,T] = \{(u,v) \in E(D) : u \in S, v \in T\}$, if $s \in S$ and $t \in T$.

capacity of cut: $cap(S,T) := \sum_{e \in [S,T]} c(e)$.

Lemma. (Weak duality) If f is a feasible flow and $[S,T]$ is a source/sink cut, then

$$val(f) \leq cap(S,T).$$

Proof.

$$\begin{aligned} cap(S,T) &= \sum_{e \in [S,T]} c(e) \\ &\geq \sum_{e \in [S,T]} f(e) \\ &\geq \sum_{e \in [S,T]} f(e) - \sum_{e \in [T,S]} f(e) \\ &= val(f). \end{aligned}$$

Max flow-Min cut Theorem

Max Flow-Min Cut Theorem (Ford-Fulkerson, 1956) Let f be a feasible flow of maximum value and $[S, T]$ be a source/sink cut of minimum capacity. Then

$$val(f) = cap(S, T).$$

Proof. (Corollary to proof of Characterization Lemma) Define

$S := \{v \in V(D) : \exists\ f$-augmenting path* from s to $v\}$.

Since f is maximum, f has no augmenting path. Then $t \in \bar{S}$ and of course $s \in S$.

$$\begin{aligned} cap(S, \bar{S}) &= \sum_{e \in [S, \bar{S}]} c(e) \\ &= \sum_{e \in [S, \bar{S}]} f(e) - \sum_{e \in [\bar{S}, S]} f(e) \\ &= val(f). \end{aligned}$$

Edge-Menger Theorem

Recall:

$\kappa'(x,y)$:= min$\{|F|$: F is an x,y-disconnecting set,$\}$
$\lambda'(x,y)$:= max$\{|\mathcal{P}|$: $\mathcal{P}$ is a set of p.e.d.* x,y-paths$\}$

* p.e.d. means pairwise edge-disjoint

Local-Edge-Menger Theorem For all $x, y \in V(G)$,

$$\kappa'(x,y) = \lambda'(x,y).$$

Proof. Build network (D, x, y, c) where $V(D) = V(G)$, $E(D) = \{(u,v),(v,u) : uv \in E(G)\}$ and $c(e) = 1$ for all $e \in E(D)$.

- 1-to-1 correspondence between x,y-disconnecting sets and sorce/sink cuts. Hence $\kappa'(x,y) = \min cap(S,\bar{S})$.
- each set of p.e.d. path determines a feasible flow. So $\lambda'(x,y) \le \max val f$.

But what if there is some clever way to direct differently a flow with **larger** overall value?? This flow then must have fractional values on some of the edges.

Ford-Fulkerson Algorithm

Initialization $f \equiv 0$
WHILE there exists an augmenting path P
    DO augment flow f along P
return f

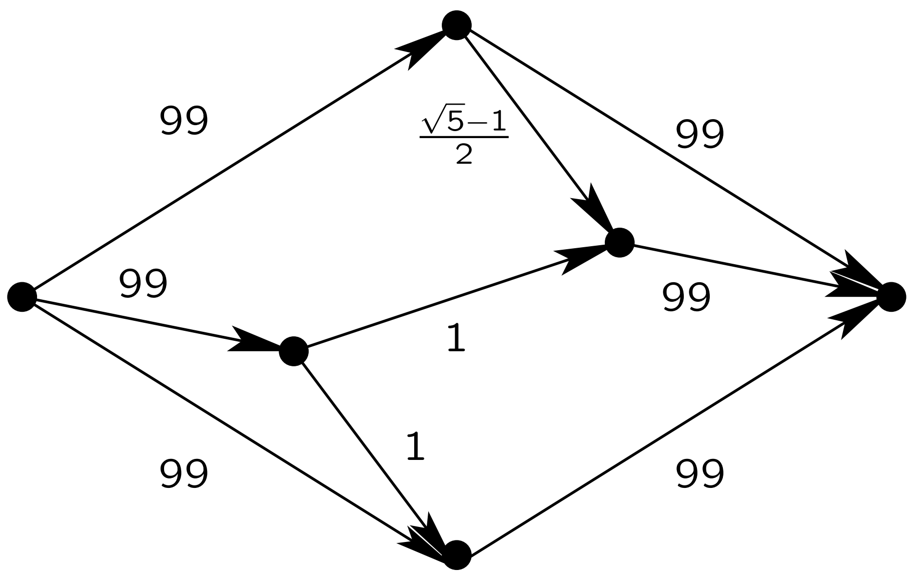
Corollary. (**Integrality Theorem**) If all capacities of a network are integers, then there is a maximum flow assigning integral flow to each edge.
Furthermore, some maximum flow can be partitioned into flows of unit value along path from source to sink.

Running times:

- Basic (careless) Ford-Fulkerson: might not even terminate, flow value might not converge to maximum;
  when capacities are integers, it terminates in time $O(m\,|f^*|)$, where f^* is a maximum flow.
- Edmonds-Karp: chooses a *shortest* augmenting path; runs in $O(nm^2)$

Example

The Max-flow Min-cut Theorem is true for real capacities as well,
BUT our algorithm might fail to find a maximum flow!!!



Example of Zwick (1995)

Remark. The max flow is 199. There is such an unfortunate choice of a sequence of augmenting paths, by which the flow value never grows above $2 + \sqrt{5}$.

Menger's Theorem

Recall:

$\kappa(x,y)$:= min$\{|S|$: S is an x,y-cut,$\}$ and
$\lambda(x,y)$:= max$\{|\mathcal{P}|$: $\mathcal{P}$ is a set of p.i.d. x,y-paths$\}$

Local-Vertex-Menger Theorem Let $x, y \in V(G)$, such that $xy \notin E(G)$. Then

$$\kappa(x,y) = \lambda(x,y).$$

Proof. We apply the Integrality Theorem for the auxiliary network (D, x^+, y^-, c).

$$\begin{aligned} V(D) &:= \{v^-, v^+ : v \in V(G)\} \\ E(D) &:= \{(u^+v^-) : uv \in E(G)\} \\ &\quad \cup\{(v^-v^+) : v \in V(G)\} \end{aligned}$$

$c(u^+v^-) = \infty^*$ and $c(v^-v^+) = 1$.

*or rather a large enough **integer**, say $|V(D)|$.

Application: Baranyai's Theorem

$\chi'(K_n) = n - 1$ is saying: $E(K_n)$ can be decomposed into pairwise disjoint perfect matchings.

k-uniform hypergraphs? $E(\mathcal{K}_n^{(k)}) = \binom{[n]}{k}$

Let $k|n$. $\mathcal{S} = \{S_1, \ldots, S_{n/k}\}$ is a "perfect matching in $\mathcal{K}_n^{(k)}$ if $S_i \cap S_j = \emptyset$ for $i \neq j$.

There is are many perfect matchings in $\mathcal{K}_n^{(k)}$.
Is there a decomposition of $\binom{[n]}{k}$ into perfect matchings?

Not obvious already for $k = 3$ (Peltesohn, 1936)
$k = 4$ (Bermond)

Theorem (Baranyai, 1973) For every $k|n$, there is a decomposition of $\binom{[n]}{k}$ into perfect matchings.

Proof of Baranyai's Theorem

Induction on the size of the underlying set $[n]$.
NOT the way you would think!!!

We imagine how the $m = \frac{n}{k}$ pairwise disjoint k-sets in each of the $M = \binom{n-1}{k-1} = \binom{n}{k}/m$ "perfect matchings" would develop as we add one by one the elements of $[n]$.

A **multi**set $\mathcal{A}$ is an m-partition of the base set X if $\mathcal{A}$ contains m pairwise disjoint sets whose union is X.

Remarks
An m-partition is a "perfect matching" in the making.
Pairwise disjoint $\Rightarrow$ only $\emptyset$ can occur more than once.

Stronger Statement For every l, $0 \leq l \leq n$ there exists M m-partitions of $[l]$, such that every set S occurs in $\binom{n-l}{k-|S|}$ m-partitions ($\emptyset$ is counted with multiplicity).

Remark For $l = n$ we obtain Baranyai's Theorem since $\binom{0}{k-|S|} = 0$ unless $|S| = k$, when its value is 1.

Proof of Stronger Statement: Induction on l.

$l = 0$: Let all $\mathcal{A}_i$ consists of m copies of $\emptyset$.

$l = 1$: Let all $\mathcal{A}_i$ consists of $m-1$ copies of $\emptyset$ and 1 copy of $\{1\}$.

Let $\mathcal{A}_1, \ldots, \mathcal{A}_M$ be a family of m-partitions of $[l]$ with the required property.
We construct one for $l+1$.

Define a network D:

$$V(D) = \{s, t\} \cup \{\mathcal{A}_i : i = 1, \ldots, M\} \cup 2^{[l]}.$$

$$E(D) = \{s\mathcal{A}_i : i \in [M]\} \cup \{\mathcal{A}_i S : S \in \mathcal{A}_i\} \cup \{St : S \in 2^{[l]}\}.$$

Edge $\mathcal{A}_i\emptyset$ has the same multiplicity as $\emptyset$ in $\mathcal{A}_i$.

Capacities: $c(s\mathcal{A}_i) = 1$

$c(\mathcal{A}_i S)$ any positive integer.

$c(St) = \binom{n-l-1}{k-|S|-1}$.

There is flow f of value M:

Flow values: $f(s\mathcal{A}_i) = 1$

$$f(\mathcal{A}_i S) = \frac{k-|S|}{n-l}$$

$$f(St) = \binom{n-l-1}{k-|S|-1}.$$

Remark. Edges of type 1 and 3 have maximum flow value.

Claim f is a flow. □

f is clearly maximum ($val(f) = cap(\{s\}, V \setminus \{s\})$).

Integrality Theorem $\Rightarrow$ there is a maximum flow g with integer values. So
$g(s\mathcal{A}_i) = f(s\mathcal{A}_i) = 1$ and
$g(St) = f(St) = \binom{n-l-1}{k-|S|-1}$.

By the conservation constraints at $\mathcal{A}_i$ there exists a unique S_i for each $i = 1, \ldots, M$ such that $g(\mathcal{A}_i S_i) = 1$.

Define m-partitions

$$\mathcal{A}'_i = \mathcal{A}_i \setminus \{S_i\} \cup \{S_i \cup \{l+1\}\}$$

of the set $[l+1]$.

Claim $\{\mathcal{A}'_1, \ldots, \mathcal{A}'_M\}$ is an appropriate family of m-partitions of $[l+1]$.

Proof. Let $T \subseteq [l+1]$.

If $l+1 \in T$, then T occurs in $\mathcal{A}'_i$ iff for $S = T \setminus \{l+1\}$ we have $g(\mathcal{A}_i S) = 1$. By conservation at vertex S:

$$|\{i \in [M] : g(\mathcal{A}_i S) = 1\}| = g(St) = \binom{n-(l+1)}{k-(|S|+1)}.$$

If $l+1 \notin T$, then T occurs in $\mathcal{A}'_i$ iff $T \in \mathcal{A}_i$ and $g(\mathcal{A}_i T) = 0$. The number of these indices i by induction and the above is equal to

$$\binom{n-l}{k-|T|} - \binom{n-(l+1)}{k-(|T|+1)} = \binom{n-(l+1)}{k-|T|}.$$

□