Diskrete Mathematik II — WS 2012 / 2013
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Exercise Sheet 12

You should try to solve and write up all the exercises. You are welcome to submit **at most** two neatly written exercises each week. You are encouraged to submit in pairs, but don't forget to mark the name of the scriber.

Exercise 1.

Let A be a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector. Prove that the following two variants of the Farkas lemma are equivalent.

(i) The system $A\mathbf{x} = \mathbf{b}$ has a nonnegative solution if and only if every $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq 0$.

(ii) The system $A\mathbf{x} \leq \mathbf{b}$ has a nonnegative solution if and only if every nonnegative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq 0$.

Exercise 2.

Let $A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$. Let $\mathbf{x}^* \in \mathbb{R}^n$ be a feasible solution of the following linear program:

$$\begin{aligned} \text{maximize } & \mathbf{c}^T\mathbf{x} \\ \text{subject to } & A\mathbf{x} \leq \mathbf{b}, \qquad (P) \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

and let $\mathbf{y}^* \in \mathbb{R}^m$ be a feasible solution of its dual:

$$\begin{aligned} \text{minimize } & \mathbf{b}^T\mathbf{y} \\ \text{subject to } & A^T\mathbf{y} \geq \mathbf{c}, \qquad (D) \\ & \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Prove that the following two claims are equivalent (this is known as *complementary slackness*):

(i) $\mathbf{x}^*$ is an optimal solution for (P) and $\mathbf{y}^*$ is an optimal solution for (D).

(ii) For every $1 \leq i \leq m$, if $\mathbf{x}^*$ does not satisfy the i-constraint of (P) with equality then $\mathbf{y}^*_i = 0$, and similarly for any $1 \leq j \leq n$, if $\mathbf{y}^*$ does not satisfy the j-constraint of (D) with equality then $\mathbf{x}^*_j = 0$.

Exercise 3.

State the dual problem D_i for the initial problem P_i where

1. (P_1)

$$\begin{aligned} \text{maximize } & z = 2x_1 + 3x_2 - x_3 + x_4 \\ \text{subject to } & x_2 - 2x_3 \leq -1 \\ & x_1 + x_2 + x_3 - x_4 \leq 0 \\ & x_1 - x_2 - 2x_3 \leq 5 \\ & x_2 - x_3 - x_4 \leq -2 \\ & x_1, \ldots, x_4 \geq 0. \end{aligned}$$

2. (P_2)

$$\begin{aligned} \text{maximize } & z = x_1 + x_2 - x_3 + x_4 - 5x_5 \\ \text{subject to } & x_i - x_{i+1} \leq 0, 1 \leq i \leq 4 \\ & x_5 - x_1 \geq 2 \\ & x_1 + x_2 + x_3 + x_4 + x_5 = -3 \\ & x_1, \ldots, x_5 \leq 0. \end{aligned}$$