Diskrete Mathematik II WS 2012 / 2013
Tibor Szabó

Exercise Sheet 13

You should try to solve and write up all the exercises. You are welcome to submit **at most** two neatly written exercises each week. You are encouraged to submit in pairs, but don't forget to mark the name of the scriber.

Exercise 1.

If G is a directed graph with n vertices and m arcs, we define a matrix $A(G)$ as follows. $A(G)$ is a $n \times m$-matrix with rows corresponding to the vertices of G, and columns corresponding to the arcs of G. The entry $A(G)_{v,e}$ on row $v \in V(G)$ and column $e \in E(G)$ is 1 if the arc e starts at v, -1 if the arc e ends at v, and 0 otherwise.

(a). Suppose C is a directed graph those underlying undirected graph is a cycle (C need not be a directed cycle). Prove that $\det(A(C))$ is 0.

(b). Prove that for any directed graph G, $A(G)$ is totally unimodular.

(c). Deduce the max-flow min-cut theorem.

Exercise 2.

Let G be a non-empty hypergraph and $0 \leq \gamma \leq 1$ real. Prove that the following two conditions are equivalent:

(i). There exists a weight function $f : V(G) \to \mathbb{R}_{\geq 0}$ satisfying $\sum_{v \in V(G)} f(v) = 1$ and $\sum_{v \in e} f(v) \geq \gamma$ for all $e \in E(G)$.

(ii). For every function $g : E(G) \to \mathbb{R}_{\geq 0}$ there is a vertex $v \in V(G)$ such that $\sum_{e : v \in e} g(e) \geq \gamma \sum_{e \in E} g(e)$.

(**Hint.** Reduce the problem to one of the equivalent forms of the Farkas lemma.)

Exercise 3.

Let $\mathcal{L}$ be a finite collection of congruent disks in the plane, such that any two have a point in common. Then there exists 5 points such that each disk of the collection contains at least one of these points.

(**Hint.** This is a problem in combinatorial geometry. To solve it, look at the set of centers of the disks. Bound the diameter of this set and then inscribe it into a square of suitable size. Now pick the 5 points in this square in such a way that any center is close to one of the points).

Exercise 4.

Let G be a graph. We define $t(G)$ as the maximum number of edge-disjoint triangles in G.

(a). Formulate an integer linear program that finds $t(G)$.

(b). By using the LP relaxation of the program at (a), describe a 3-approximation algorithm for finding the value of $t(G)$.