Diskrete Mathematik II WS 2012 / 2013

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Exercise Sheet 9

Due date: Dec 18, 12:30 PM, beginning of exercises
NO LATE SUBMISSIONS!

You should try to solve and write up all the exercises. You are welcome to submit **at most** two neatly written exercises each week. You are encouraged to submit in pairs, but don't forget to mark the name of the scriber.

Exercise 1.

Consider the following greedy algorithm for solving the minimum cover problem for a graph G.

At every step, we pick a vertex not chosen before, that covers the maximum possible number of still uncovered edges, and we add it to our cover. This results in a subset M of vertices of G which cover all the edges of G.

Construct a graph G for which the result M of the greedy algorithm is at least 100 times larger than the optimal solution. Argue that this is indeed the case.

(**Hint**: Consider a bipartite graph with vertex classes of sizes $k2^{k-1}$ and 2^k, respectively).

Exercise 2.

Recall that the matching number $\alpha'(G)$ of a graph G is the size of a largest matching in G.

a) Formulate an integer linear program that finds the matching number of a graph, and where the variables correspond to the edges of G.

b) Formulate the LP relaxation of the integer program defined at a). Denote its optimal solution by $\alpha'_f(G)$.

c) Show that for any graph G there exists an optimal solution of the LP relaxation defined at b), where each variable has value $0, 1/2$ or 1. Deduce that $2\alpha'_f(G)$ is always an integer.

(**Hint**: Among all optimal solutions, choose one having as many variables equal to 0 as possible. Now consider the structure of the subgraph induced by the edges $e \in E(G)$ having the corresponding variable non-zero).

Exercise 3.

Transform the following linear program to equational form.

minimize $5x_1 - 6x_2 + 3x_3$

subject to

$$
\begin{aligned}
3x_1 + x_2 &\leq -1 \\
13x_1 + 7x_2 &\geq 1 \\
2x_1 + 5x_3 &\leq 0 \\
x_1 + x_2 + x_3 &\geq 1 \\
x_1 &\leq 0 \\
x_2 &\geq 0
\end{aligned}
$$

Exercise 4.

We consider a linear program in equational form

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0},$$

with $\mathbf{c} \in \mathbf{R}^n, \mathbf{b} \in \mathbf{R}^m, A \in \mathbb{R}^{m\times n}$ and $\operatorname{rank}(A) = m$.

As discussed in the lecture if this linear program is bounded, there exists a finite, but inefficient, algorithm for finding a solution. Consider all m-element subsets $B \subseteq \{1, 2, \ldots, n\}$ and for each of them decide if B is a feasible basis, by solving a system of linear equations. Then take the maximum of the objective function over all basic feasible solutions found in this way.

Explain how to modify this algorithm so that it also works for unbounded linear programs (in this case, it must report that the linear program is unbounded). Prove that the modified algorithm is correct and terminates. Be particularly precise in proving correctness.

(**Hint.** If the worst-case time complexity of the initial algorithm is $f(n, m)$, then it is possible to modify it in such a way that the worst-case time complexity of the new algorithm is not larger than $O(f(2n, n + m))$).