7 The Szemerédi–Trotter theorem

In an abrupt change of topic we now move to combinatorial geometry.

We wish to derive sharp estimates on the number of incidences between a finite set of lines and a finite set of points in the plane $\mathbb{R}^2$. More on this later on.

It turns out that in the proof we will need the notion of a drawing of a graph and of the crossing number of a graph.

Definition. Let be G a finite graph. A *drawing* is a map that takes vertices to points of $\mathbb{R}^2$ and edges to curves (smooth functions from $[0, 1] \mapsto \mathbb{R}^2$) which start and end at the images of their endpoint-vertices.

Example. Let $G = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{4, 5\}\}$. Give two genuinely different drawings.

Definition. Let be G a finite graph. The *crossing number* of G is the least number of crossings in any drawing of G on the plane (the number of points where a pair of edges intersect, excluding intersections at vertices).

Let us consider G with vertex set $V = \{1, 2, 3, 4\}$ and edge set $E = \{\{1, 2\}, \{3, 4\}\}$.

$\mathrm{cr}(G) = \quad .$

Real time exercise. For each of the following cycles determine the crossing number.

(i) C_3 with vertex set $V = \{1, 2, 3\}$ and edge set $E = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$.

(ii) C_4 with vertex set $V = \{1, 2, 3, 4\}$ and edge set $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$.

Let us now quickly cover some facts about the so-called planar graphs.

Definition. A graph is called *planar* if its crossing number is zero.

For example, the 5-cycle C_5 is a planar graph, yet the complete graph on 5 vertices K_5 is not.

When a graph is planar, one can talk about its faces.

A *face* of a finite planar graph with at least three vertices is a connected region of the subset of the plane resulting from removing a drawing of the graph.

In the examples we saw above the faces are:

A word of warning: an intuitive definition of the word 'face' result in one fewer face than the definition. The definition forces a triangle to have two faces. This is true in general. All finite planar graphs have an unbounded face, which comes from the unbounded component.

Euler proved that for finite connected planar graphs.

$$|F| - |E| + |V| = 2.$$

Here $|F|$ is the number of faces (and as usual $|E|$ the number of edges and $|V|$ the number of vertices).

Let us deduce an upper bound on the number of edges in a finite planar graph, which will be useful later on. The controversial faces will not appear.

Lemma 7.1. *Let $G = (V, E)$ be a finite planar graph. Then $|E| < 3|V|$.*

Proof.

□

Corollary 7.2. *Let $G = (V, E)$ be a finite graph. Then $|E| < 3|V| + \mathrm{cr}(G)$.*

Proof.

$\square$

We now prove a lower bound on the number of crossings.

Lemma 7.3 (Ajtai–Chvátal–Newborn–Szemerédi, Leighton)**.** *Let $G = (V, E)$ be a finite graph. Suppose $|E| \geq 4|V|$. Then*

$$\mathrm{cr}(G) \geq \frac{|E|^3}{64|V|^2}.$$

Proof.

$\square$

Remark. The bound is attained up to a constant on K_n, the complete graph on n vertices.

We use this inequality to establish an upper bound on the number of point-line incidences.

Definition. Let P be a finite set of points and L be a finite set of lines on the plane $\mathbb{R}^2$. The number of point-line incidences is

$$I(P, L) = |\{(p, \ell) : p \in P, \ell \in L, p \in \ell\}|.$$

Examples.

Theorem 7.4 (Szemerédi–Trotter)**.** *Let P be a finite set of points and L be a finite set of lines on the plane $\mathbb{R}^2$. Then number of point-line incidences is at ost*

$$I(P, L) \leq 4(|L|^{2/3}|P|^{2/3} + |L| + |P|).$$

Proof by Székely.

$\square$

Remark. All three terms are necessary.

The Szemrédi–Trotter theorem has plenty of applications in unexpected places. The most spectacular has to be that of Elekes on the sum-product problem of Erdős.

Conjecture. *Let $A \subset \mathbb{R}$ be a finite set of real numbers. Then*

$$\max\{|A \cdot A|, |A + A|\} \gg |A|^{2-\varepsilon} \text{ for all } \varepsilon > 0.$$

Remark. The ε is necessary. For example when $A = \{1, \ldots, n\}$, then $|A \cdot A| \ll |A|^2 / \log\log(n)$. This is a non-trivial statement. It can be deduced by applying Chebyshev's inequality to the function $\omega(i)$ that counts the number of distinct prime factors of the positive integer i. A result of Erdős and Kac states that $\omega(n)$ (viewed as a random variable under the uniform distribution on $\{1, \ldots, n\}$) has asymptotic mean $\log\log(n)$ and asymptotic variance $\sigma^2 = \log\log(n)$.

Theorem 7.5 (Elekes)**.** *Let $A \subset \mathbb{R}$ be a finite set of real numbers. Then*

$$|A \cdot A|\,|A + A| \geq \frac{|A|^{5/2}}{16}.$$

In particular

$$\max\{|A \cdot A|, |A + A|\} \geq \frac{|A|^{5/4}}{4}.$$

Proof.

$\square$