Problems for Week I

Note: Starred exercises are additional, in the sense that they are not as central to the course. They are usually longer, though not necessarily harder than the rest.

Problem 1. Give a rigorous proof of Ruzsa's triangle inequality: $|A||B-C| \leq |B-A||A-C|$.

Problem 2. Another proof of Ruzsa's triangle inequality.

Recall that the *convolution* of two functions f and g (with discrete support) is defined as

$$f * g(x) = \sum_z f(z)g(x-z).$$

Prove the triangle inequality by comparing a lower and an upper bound for the quantity

$$Q = \sum_{v \in B-C} 1_{B-A} * 1_{A-C}(v).$$

As usual, 1_S is the *characteristic function* of a set S defined as $f(x) = \begin{cases} 1\,, & x \in S \\ 0\,, x & \notin S \end{cases}$.

Problem 3*. An explanation of the term 'triangle inequality'. Let X, Y and Z be finite non-empty sets in a group. Let

$$d(X,Y) = \log\left(\frac{|X-Y|}{|X|^{1/2}|Y|^{1/2}}\right).$$

Prove that $d(X,Z) \leq d(X,Y) + d(Y,Z)$. Is d a metric (on the set of finite non-empty subsets of the commutative group)?

Problem 4. Let $n \geq 4$ be a positive integer.

(i) Suppose that $|A+A-A| \leq \alpha|A|$. Find an upper bound for the cardinality of the n-fold sumset $A+A-A+\cdots+(-1)^n A$ in terms of α and $|A|$.

(ii) Suppose that $|A+A+A| \leq \beta|A|$. Find an upper bound for the cardinality of the n-fold sumset $A+A-A+\cdots+(-1)^n A$ in terms of β and $|A|$.

Problem 5. Let A and B be finite non-empty sets of a commutative group. We have seen the importance of the subset X that minimises the quantity $\dfrac{|Z+B|}{|Z|}$ over all non-empty subsets Z of A. In each of the following three examples identify X.

(i) A is a subgroup and B is any non-empty set of the ambient group.

(ii) $A = B$ is an arithmetic progression in $\mathbb{Z}$.

(iii) $A = B$ is the subset of $\mathbb{Z}^3$ that consists of the union of the "discrete cube" $\{(x, y, z) : 1 \leq x, y, z \leq n\}$ with three "elongated edges" $\{(x, 0, 0) : 1 \leq x \leq n^2\}$, $\{(0, y, 0) : 1 \leq y \leq n^2\}$ and $\{(0, 0, z) : 1 \leq z \leq n^2\}$.

Problem 6*. Let A, B and C be finite non-empty sets of a commutative group and X the minimiser associated with A and B described in the problem above. Here is a proof of the inequality $|X|\,|X + B + C| \leq |X + B|\,|X + C|$ that was given by Reiher.

(i) Use Hall's marriage theorem a.k.a. König's theorem to prove the existence of a bijection ϕ from $X \times (X + B)$ to itself with the special property that if $\phi(r, s) = (x, y)$, then $y - r \in B$.

(ii) Use ϕ to construct an injection $\theta : X \times (X + B + C) \mapsto (X + B) \times (X + C)$ as follows:

- order the elements of C in some way;

- write each $s \in X + B + C$ as the sum $t + c$, where $t \in X + B$ and $c \in C$ is minimal in the chosen order;

- map $(a, s) \in X \times (X + B + C)$ to $(y, x + c)$, where $s = t + c$ as above and $(x, y) = \phi^{-1}(a, t)$.

Problem 7*. Can you prove the inequality $|A|\,|B + C| \leq |A + B|\,|A + C|$ in a similar way to the triangle inequality?

Problem 8. Plünnecke's inequality for a large subset. Let $\varepsilon > 0$. We prove that there exists $\emptyset \neq Y \subseteq A$ of cardinality at least $(1 - \varepsilon)|A|$ such that $|Y + hA| \leq (\alpha/\varepsilon)^h |A|$. Fill in the details in the following steps.

Apply Plünnecke's inequality to the pair (A, A) to find $\emptyset \neq X_1 \subseteq A$ such that $|X_1 + hA| \leq \alpha^h |X_1|$. If X_1 is large enough we are done.

Otherwise apply Plünnecke's inequality to the pair $(A \setminus X_1, A)$ to find $\emptyset \neq X_2 \subseteq A \setminus X_1$ such that $|X_1 + hA|$ is bounded in a reasonable way (your job is to find a reasonable bound).

Keep going this way until $|X_1| + \cdots + |X_k| > (1 - \varepsilon)|A|$.

One cannot take $Y = A$ in general. Can you find an example?

Problem 9. Plünnecke's inequality and one of the many inequalities of Ruzsa we have seen imply that if $|A+A| \leq \alpha|A|$, then

$$|A+A+A| \leq \min\{\alpha^3|A|, \alpha^{3/2}|A|^{3/2}\}.$$

Modify the example given in Problem 5 (iii) to show that the bound is up to constant sharp (for infinitely many values of α that form an unbounded sequence).

Problem 10. A covering lemma of Green and Ruzsa. Let A and B be finite non-empty sets in a commutative group. Suppose that $|A+B| \leq \alpha|A|$. Prove there exists a set S of cardinality at most 2α such that every element $b \in B$ can be expressed in at least $|A|/2$ ways as a sum $b = s + a - a'$ where $a, a' \in A$ and $s \in S$, i.e. prove that for all $b \in B$

$$|\{(s, a, a') \in S \times A \times A : b = s + a - a'\}| \geq \frac{|A|}{2}.$$

Problem 11*. For those that like the probabilistic method. We prove that with high probability $\log(n)$ translates of a random subset of $\{1, \ldots, n\}$ are not adequate to cover the entire set.

A random subset of $\{1, \ldots, n\}$ is formed by including each element uniformly with probability $1/2$ independently of all the others. Let A be such a random set and $S \subseteq \{1, \ldots, n\}$ be a set of cardinality $\log(n)$.

(i) Find the probability that $i \in \{1, \ldots, n\}$ belongs to $S + A$.

(ii) Deduce an upper bound for the probability that $S+A$ covers $\{1, \ldots, n\}$ i.e., that $\{1, \ldots, n\} \subseteq A + S$.

(iii) Use a so-called union bound to prove that with probability $1 - o(1)$ there is no set S of cardinality $\log(n)$ such that $S + A$ covers $\{1, \ldots, n\}$.

We have therefore established that with high probability, $\log(n)$ translates of a random set do not cover $\{1, \ldots, n\}$. This is an existence proof. We know that most subsets of $\{1, \ldots, n\}$ have this property, but our proof does not provide us with an explicit example.

Problem 12*. Use Problem 11 (as a black box, if necessary) to prove that for, say, all $\alpha > 2$ there exist sets A and B such that $|A+B| \leq \alpha|A|$ and B cannot be covered with fewer than $c\log(|A|)\alpha$ translates of A, where c is an absolute constant.

Problem 13. Let A be finite non-empty set in a commutative group. Suppose that $|A+A| \leq \alpha|A|$. Here is another proof of the bound $|A+A+A| \leq \alpha^3|A|$ in four steps:

(i) Find an $\emptyset \neq X \subseteq A$ such that both $|X+A|$ and $|X+A+A|$ are "small".

(ii) Cover A by translates of X.

(iii) Deduce a covering for $A+A+A$ by translates of $X+A+A$.

(iv) Apply the power trick.

Problem 14. Let A, B_1 and B_2 be finite non-empty sets in a commutative group. Suppose that $|A+B_i| \leq \alpha_i|A|$ for $\alpha_i \in \mathbb{Q}$. In this exercise we establish the bound

$$|B_1+B_2| \leq \alpha_1\alpha_2|A|.$$

We saw that if $\alpha_1 = \alpha_2 = \alpha$, then $|B_1+B_2| \leq \alpha^2|A|$. Let us now deduce the general case from this.

Let n_1 and n_2 be positive integers such that $n_1\alpha_1 = n_2\alpha_2 \in \mathbb{Z}$. Why do such n_i exist? Work in the direct product of the ambient group with $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. Apply the above result to $A' = A \times \{0\} \times \{0\}$, $B_1' = B_1 \times Z_{n_1} \times \{0\}$ and $B_2' = B_2 \times \{0\} \times Z_{n_2}$.

Problem 15. For each of the following provide a proof or counter example.

(i) Every Freiman homomorphism is a homomorphism between the ambient groups.

(ii) Every homomorphism between the ambient groups is a Freiman isomorphism between a set and its image.

Problem 16. For each of the following you are given a finite set A and a positive integer k. Find a set $B \subset \mathbb{Z}$ and an explicit formula for a Freiman k-isomorphism from A to B.

(i) $A = \{(x,y) \in \mathbb{Z}^2 : 0 \leq x,y \leq n-1\}$, is the discrete square of side length n and $k=2$.

(ii) $A = \{(x,y) \in \mathbb{Z}^2 : 0 \leq x,y \leq n-1\}$, is the discrete square of side length n and $k=n$.

(ii) $A = \{(x_1,\ldots,x_d) \in \mathbb{Z}^d : 0 \leq x_1,\ldots,x_n \leq n-1\}$, is the discrete d-dimensional cube of side length n and $k=n^2$.

Problem 17. Let A and B be two sets in a commutative group that contain 0. Suppose that $\theta : A+A \mapsto B+B$ is a k-Freiman isomorphism that satisfies $\theta(0)=0$. Prove that A is $(2k)$-Freiman isomorphic to B.

Problem 18*. Is there a non-trivial Freiman isomorphism from the unit circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ to a subset of $\mathbb{R}$ that is a continuous function (on $\mathbb{T}$)?

Problem 19. For each of the following you are given a set of integers and positive integers k. Go through the proof of Ruzsa's Freiman-isomorhism theorem and construct a k-Freiman isomorphic subset of $\mathbb{Z}_p$ for a suitable p.

(i) $A = \{1, 3, 5, \ldots, 17\}$ and $k = 2$.

(ii) $A = \{2, 3, 5, 8\}$ and $k = 4$.

Problem 20. For each of the following sets get an exact formula for the required quantity.

(i) The additive energy of an arithmetic progression of length n.

(ii) The additive energy of a Sidon set (a set where the sums $x + y$ are distinct) of cardinality n.

(iii) The expected value of the additive energy of an random set in $\mathbb{Z}/p\mathbb{Z}$ where each element appears with probability p.

Problem 21. We have seen that small doubling implies large additive energy and that large additive energy implies small doubling for a large subset.

Estimate the doubling constant and the additive energy of the set

$$A = \{1, \ldots, n\} \cup \{n^2, n^3, \ldots, n^n\}.$$

What does this example tell us?

Problem 22. The Balog-Szemerédi-Gowers theorem continues to hold when "addition takes place along the edges of a graph $G \subseteq A \times B$".

Let A and B be sets in a commutative group. Define $A +_G B = \{a + b : (a, b) \in G\}$.

(i) Find $A +_G B$ when $A = B = \{1, \ldots, 2n\} \subset \mathbb{Z}$ and $G = \{1, \ldots, n\} \times \{1, \ldots, n\} \subset \mathbb{Z}^2$.

(ii) Prove that $E(A, B) \geq \dfrac{|G|^2}{|A +_G B|}$.

(iii) Prove that if $E(A, B) \geq \dfrac{|A|^{3/2}|B|^{3/2}}{K}$, the there exists $G \subset A \times B$ of density at least $1/K$ that satisfies $|A +_G B| \leq \dfrac{K|G|}{\sqrt{|A|\,|B|}}$.

Problem 23. Find the exact number of point-line incidences when $P = \{1, \ldots, n\} \times \{1, \ldots, 2n^2\}$ and L is the set of lines $y = ax + b$ where $(a, b) \in \{1, \ldots, n\} \times \{1, \ldots, n^2\}$.

What does this example show?

Problem 24. We establish the so-called Cauchy-Schartz lower bound on the number of point-line incidences. Let L be a finite collection of lines and P be a finite set of points in the plane $\mathbb{R}^2$. For each $\ell \in L$ let n_ℓ denote the number of points from P that are on ℓ.

Complete each of the following steps.

(i) Evaluate $\sum_{\ell \in L} n_\ell$.

(ii) Deduce $\sum_{\ell \in L} (n_\ell - 1)$.

(iii) Explain why $\sum_{\ell \in L} \binom{n_\ell}{2} \leq \binom{|P|}{2}$ is true.

(iv) Deduce an upper bound for $\sum_{\ell \in L} (n_\ell - 1)^2$.

(v) Finally prove

$$I(P, L) \leq |L| + |L|^{1/2}|P|.$$

Problem 25*. Formulate and prove an analogue to the Szemerédi-Trotter theorem for the number of incidences between a finite set of points P and a finite collection of circles of equal radii C in the plane.