Algorithmic Combinatorics WS 2014 / 2015
Tibor Szabó
Shagnik Das

Mock Exam

Due date: Jan 16th, 12:00 noon, tutor box of Shagnik Das.

This mock exam is optional, and is provided for your benefit. You will have three hours to solve the real exam, without notes or texts or discussion with any other sentient being. You may wish to solve this mock exam under these exam conditions, but this is not required. Even if you do work with others on these problems, submit your solutions **individually**. The exams will be graded and returned in the following exercise classes.

Each problem below is worth 10 points. You are not allowed to use results from homework or exercise classes without proof.

1. Let $\alpha \in \mathbb{R}$, and let A, B be finite non-empty subsets of a commutative group with $|A+B| \leq \alpha |A|$. Show there is some set $S \subset B$ with $|S| \leq \lfloor \alpha \rfloor$ such that $B \subseteq S + A - A$.

2. Let $X := \{f, g, h, i, j\}$ and $Y := \{a, b, c, d, e\}$ and consider the complete bipartite graph G consisting of all edges between X and Y. We assign weights to the edges of G according to the following matrix, where the lines correspond to vertices in X, and the columns to vertices in Y:

$$\begin{array}{c} \\ f \\ g \\ h \\ i \\ j \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 7 & 8 & 9 & 8 & 7 \\ 8 & 7 & 6 & 7 & 6 \\ 9 & 6 & 5 & 4 & 6 \\ 8 & 5 & 7 & 6 & 4 \\ 7 & 6 & 5 & 5 & 5 \end{pmatrix} \end{array}$$

Find a maximum-weight matching in G using the Hungarian algorithm, and give a short proof that your matching is optimal.

3. (a) Define the list-chromatic number $\chi_\ell(G)$ of a graph G.

(b) Show that $\chi_\ell(K_{t,s}) \geq t + 1$ if $s \geq t^t$.

4. (a) Describe the Proposal Algorithm, by specifying the input, the iteration step, and the output of the algorithm.

(b) Show that the Proposal Algorithm with the men proposing gives the worst possible outcome for the women; that is, prove there is no stable matching where one of the women is paired with a worse partner than in the matching produced by the Proposal Algorithm.

5. Show that for every 3-regular graph G, $\kappa(G) = \kappa'(G)$.

6. Let $G = (V, E)$ be a simple undirected graph and let $\mathcal{I} \subseteq 2^V$ be the family of all independent sets of G. Consider the following integer linear program. We have a variable x_I for each independent set $I \in \mathcal{I}$. We want

$$\min \sum_{I \in \mathcal{I}} x_I,$$

subject to the constraints that $x_I \in \mathbb{Z}, x_I \geq 0$ for every $I \in \mathcal{I}$ and for every vertex $v \in V$, $\sum_{I \ni v} x_I \geq 1$.

(a) Name the graph parameter which is equal to the minimum value of the objective function.

(b) Prove properly that the two are equal.

(c) Give, with justification, a graph for which the LP relaxation of the integer linear program has a different value.