Algorithmic Combinatorics WS 2014 / 2015
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Exercise Sheet 4

Due date: Nov 14th, 2:00 PM, tutor box of Shagnik Das
Late submissions will not be tolerated!

You should try to solve and write up all the exercises. You are welcome to submit **at most** three neatly written exercises for correction each week. You are encouraged to submit in pairs, but please indicate the author of each solution. Each problem is worth 10 points.

Exercise 1. Consider the following algorithm to find the minimum of a set of 2^k numbers.

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Algorithm: MIN
Data: A = {a_1, ..., a_n}, n = 2^k ≥ 2
Result: MIN(A) = min{a_1, ..., a_n}
if n = 2 then
    if a_1 < a_2 then
        return a_1;
    else
        return a_2;
    end
else
    for 1 ≤ i ≤ n/2 do
        set y_i = MIN({x_{2i-1}, x_{2i}});
    end
    return MIN({y_1, ..., y_{n/2}});
end
```

(i) Show that the MIN algorithm requires $n-1$ comparisons to find the minimum element, and that this is the best possible.

(ii) After running the MIN algorithm to find the minimal element, how many additional comparisons are required to find the second-smallest element?

(iii) Deduce a sorting algorithm that requires $(1+o(1))n\log_2 n$ comparisons to sort $n = 2^k$ elements.

Exercise 2. You know that among a stack of n coins, there is one counterfeit coin. Furthermore, you know the counterfeit coin is lighter than the genuine coins. You have a balance that, given two disjoint sets of coins A and B, can tell you if the coins in A are lighter than, heavier than or the same weight as those in B. How many weighings does it take to find the counterfeit coin? Give a lower bound and an algorithm which achieves this.

Bonus (5 pts): What if the counterfeit coin could be lighter or heavier?

Exercise 3. Let us play a game. I think of an integer x between 1 and n, and your job is to try and determine x. You are allowed to ask questions of the form "Is $x < y$?" or "Is $x > y$?" for any y.

(i) Show that you can find x with only $\lceil \log_2 n \rceil$ questions, and that this is best possible.

To make your job slightly harder, I am allowed to lie to you k times.

(ii) Show that you can find x with at most $(2k+1)\lceil \log_2 n \rceil$ questions.

(iii) Show that you can find x with at most $(k+1)\lceil \log_2 n \rceil + k^2$ questions.

Bonus (up to 10 pts for the best answer): Can you do even better? How few questions are needed with k lies?

Exercise 4. Show that the first k edges added in Kruskal's algorithm form a k-edge forest of minimum weight.

Exercise 5. Let $G = (V, E)$ be a weighted graph with no two edges having the same weight. For every vertex $v \in V(G)$, let $e_v \in E_G$ be the edge containing v of minimum weight, and let $E_0 = \{e_v : v \in V(G)\} \subseteq E$ be the set of all such edges. Show that *every* minimum spanning tree in G must contain all edges in E_0.

Exercise 6. Given a weighted graph G with m edges, order the edges by weight so that $w(e_1) \geq w(e_2) \geq \ldots \geq w(e_m)$. Set $G_0 = G$. At step $1 \leq t \leq m$, if the graph $G_{t-1} - e_t$ is connected, delete e_t, and set $G_t = G_{t-1} - e_t$. Otherwise keep e_t and set $G_t = G_{t-1}$.

Show that the final graph G_m is a minimum weight spanning tree.

Exercise 7. Show that every bipartite graph $G = (X \cup Y, E)$ contains a subset $S \subseteq X$ such that $\alpha'(G) = |X| - |S| + |N(S)|$.