Algorithmic Combinatorics WS 2014 / 2015

Tibor Szabó

Shagnik Das

Exercise Sheet 10

Due date: Jan 9th 2:00 PM, tutor box of Shagnik Das
Late submissions will be abandoned faster than my new year's resolutions.

You should try to solve and write up all the exercises. You are welcome to submit **at most** three neatly written exercises for correction each week. You are encouraged to submit in pairs, but please indicate the author of each solution. Each problem is worth 10 points.

Exercise 1. A print shop has three different printers, P_1, P_2 and P_3, each of which can print in black-and-white or colour. When printing in black-and-white, printer P_i can print b_i pages per minute, and when printing in colour, printer P_i can print c_i pages per minute.

Demand for printing jobs is reasonably low on Christmas Eve, and so the only staff member in attendance is the young Scottish employee, McPaper, who, in lieu of a monetary annual bonus, was recently promoted to the position of Executive Secretary to the Associate Deputy Printer Manager. In the morning, he receives a list of n jobs, $J_1, J_2, \ldots, J_n$. The job J_j requires r_j black-and-white pages and s_j colour pages to be printed.

McPaper has to divide the printing work between the machines, telling[1] each printer which pages it should print. As soon as the final page has been printed, he can shut up shop and race to the stores to complete his Christmas shopping. Formulate an integer programming problem to determine how he should divide the work to maximise the time he has left for his shopping, and justify the correctness of your problem.

Bonus (5 pts): Explain how he can find an efficient estimate to the optimal schedule, and prove that this estimated solution requires at worst an extra Cn minutes, where C is a constant depending only on the parameters b_i and c_i.

[1] His remarkable ability to communicate with the otherwise inanimate pieces of technology earned him the nickname 'The Printer Whisperer'.

Exercise 2. Having optimally scheduled the printing jobs from Exercise 1, McPaper rushes to the store to buy Christmas presents for his family. His first stop, naturally, is the game store, for while he loves his parents dearly, he most enjoys giving gifts to his youngest brother[2]. Much to his relief, he finds one copy of this year's hottest game[3], *Interstellar*, partially hidden on a shelf behind an Xbox One. Clutching the handheld video game gratefully to his chest, he reads the description of the game printed on the back of the box.

> Welcome to the wonderful world of *Interstellar*! The year is 2048, and you are the state-of-the-art robot spaceship pilot, codename TARS. Having travelled through space, time, and even a wormhole, you arrive at your destination planet.
>
> You begin at a height of 10 kilometres above the surface of the planet, descending at a speed of 100 m/s. The planet's gravitational field causes a downward acceleration of 17 m/s^2. To slow your descent you must use your engine; if you send x litres of fuel to your engine, it will create an upward acceleration of $2x$ m/s^2. The engine can handle an input of at most 10 litres of fuel. You start with 1000 litres of fuel in your tanks.
>
> However, due to the unforeseen relativistic aftereffects of travelling through a wormhole[4], time has become discrete, so the state of your spaceship only changes every second.
>
> Your human overlords order you to make the spaceship land on the planet's surface after exactly 150 seconds[5]. They remind you that, to protect the sensitive equipment on board, the spaceship's speed should be at most 5 m/s when making contact with the surface.
>
> Can you control the flow of fuel to the engine and land the spaceship safely without running out of fuel? The fate of humanity lies in your cold, metallic hands.

McPaper laughs gleefully, thinking of the many hours of fun his younger brother will have, trying to beat this game by trial-and-error. He, of course, knows that you can find the solution using linear programming. Set up a linear program to solve this game, and explain all the variables and constraints involved.

[2]The middle brother he is currently not so fond of – fickle are fraternal friendships.

[3]Produced by a prominent toy company whose name cannot be revealed, a highly successful advertising campaign led to record sales.

[4]That's relativity, folks!

[5]For reasons beyond the creativity capacities of the author of this problem.

Exercise 3. As McPaper pockets the receipt for the \$250 he paid for *Interstellar*, he curses the greed of the corporate fat cats at the unnamed prestigious toy company. His unhappiness, though understandable, is perhaps unfair, as he is unaware of quite how many people are involved in the production of the game. One of those people is Professor Professor, the mathematically gifted but ethically questionable professor who developed the physics engine behind the game.

Professor Professor is currently marking the final exams for her course[6] "Politicizing Beyoncé". The exam has 5 questions, Q_1, Q_2, Q_3, Q_4 and Q_5. There are 11 students in the course, $S_1, S_2, \ldots, S_{11}$. Student S_i gets a percentage of $p_{i,j}$ on question Q_j.

However, Professor Professor has a trick up her sleeve – she has never said that all the questions will be equally weighted for the final grade. She is free to choose weights w_j, $1 \leq j \leq 5$, that she will use to average the marks from each question for the final grade. A student passes the course if he or she has a final grade of at least 50%.

There is a set $\mathcal{L} \subset \{S_1, \ldots, S_{11}\}$ of students that Professor Professor likes, and she wants to make sure that they pass the course. On the other hand, there is a disjoint set $\mathcal{D} \subset \{S_1, \ldots, S_{11}\}$ of students she dislikes, who have caused her nothing but trouble throughout the semester. She wants them all to fail. Set up, with explanation, a linear programming problem to help her find weights w_j to achieve her goals.

Bonus (5 pts): After switching off her Beyoncé album, and listening to some Christmas carols instead, Professor Professor is in a much better mood, and decides to forgive the students in $\mathcal{D}$. She now wishes to choose weights to pass as many students as possible. Find a linear programming problem with a mix of real and integer variables to find suitable weights.

Exercise 4. With her grading finished and the semester over, Professor Professor is able to go back to her other principal duty as an academic – research. She thinks about her favourite open problem - the P vs NP problem. Inspired by her Beyoncé tunes playing in the background, she finds a polynomial-time algorithm to solve an NP-complete problem. She quickly writes up her findings in a paper, and then travels to the Clay Institute to pick up her million-dollar prize for having solved this previously-open Millenium Problem.

Now rich beyond her wildest dreams, she resigns from the university to pursue her true calling in life – the ownership and management of a football team[7]. However, when she calls Pep Guardiola, asking if she can buy Manuel Neuer, he laughs at her, offering some rather rude suggestions about what she can do with her cash. Realising that she cannot afford to buy a serious team with just a million dollars, she goes back to her apartment.

The End.

Just kidding. Winners never quit. She solves the remaining five open Millenium Prob-

[6] This course really exists: `http://www.politicizingbeyonce.com/about.html`.

[7] After all, someone will have to replace Dortmund in the Bundesliga.

lems[8], winning the Abel prize, the Shaw prize, the Fields medal and the Breakthrough prize for her efforts. Just for good measure, she also picks up three Nobel prizes[9]. Armed with her total prize money of 14.615 million dollars, she returns to the football transfer market, ready to do business. No one laughs at her now.

There are a number of players available on the market. There are goalkeepers $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, defenders $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$, midfielders $\mathcal{M} = \{M_1, M_2, \ldots, M_n\}$ and forwards $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$. Let $\mathcal{P} = \mathcal{G} \cup \mathcal{D} \cup \mathcal{M} \cup \mathcal{F}$ denote the total set of players. The function $p : \mathcal{P} \to \mathbb{R}$ gives the prices of the players (in millions of dollars).

To form an eligible team, Professor Professor must buy a total of 11 players, with exactly 1 goalkeeper, at least 3 defenders, at least 3 midfielders and at least 2 forwards.

(i) Set up an integer programming problem to determine whether she can afford to buy an eligible team, explaining all the variables and constraints.

(ii) Can one find a solution using linear programming? An approximation?

Bonus (5 pts): Professor Professor knows that you cannot win just by throwing money at players – you have to throw money at the right players! A second function $s : \mathcal{P} \to \mathbb{R}$ measures the skill of each player, and the total skill of the team is the sum of the skills of each of the players. Professor Professor wants to buy the most skillful team she can afford. Set up the corresponding linear program. How well does it approximate the solution to the integer programming problem?

Bonus Exercise (5 pts). Write an accurate and original[10] word problem to accompany the linear program below.

$$\text{minimise } 20\sum_{i=1}^{12}(u_i + d_i) \text{ subject to } \begin{cases} u_i, d_i \geq 0 \text{ for } 1 \leq i \leq 12, \\ t_0 \in \mathbb{R}, \\ t_i = t_{i-1} + u_i - d_i \text{ for } 1 \leq i \leq 12, \\ t_i - \left(10 - 15\cos\frac{i\pi}{6}\right) \leq 2 \text{ for } 1 \leq i \leq 12, \\ \left(10 - 15\cos\frac{i\pi}{6}\right) - t_i \leq 2 \text{ for } 1 \leq i \leq 12, \text{ and} \\ t_{12} = t_0. \end{cases}$$

[8]This part of the problem is fictional. At the time of writing, there are still six open Millenium Problems, and you are welcome to work on them.

[9]Economics, because the Travelling Salesman Problem is no longer a problem. Literature, because of the tremendous style with which her paper was written. Peace, because now the Stable Roommate problem is feasible, making domestic disputes a thing of the past.

[10]And preferably, if not necessarily, funny!