Algorithmic Combinatorics — WS 2014 / 2015

Tibor Szabó
Shagnik Das

Exercise Sheet 11

Due date: Jan 23rd, 2:00 PM, tutor box of Shagnik Das
Late submissions will feed my fireplace[1].

You should try to solve and write up all the exercises. You are welcome to submit **at most** three neatly written exercises for correction each week. You are encouraged to submit in pairs, but please indicate the author of each solution. Each problem is worth 10 points.

Exercise 1. In class we proved the duality theorem for primal problems of the following form: $\max \mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Using this result, find the dual of the following linear program, and prove the corresponding duality theorem.

$$\begin{aligned} \max \mathbf{c}_1^T\mathbf{x}_1 + \mathbf{c}_2^T\mathbf{x}_2 \text{ subject to} \\ \mathbf{x}_1 \geq \mathbf{0} \\ A^{(1,1)}\mathbf{x}_1 + A^{(2,1)}\mathbf{x}_2 \leq \mathbf{b}_1 \\ A^{(1,2)}\mathbf{x}_1 + A^{(2,2)}\mathbf{x}_2 \geq \mathbf{b}_2 \\ A^{(1,3)}\mathbf{x}_1 + A^{(2,3)}\mathbf{x}_2 = \mathbf{b}_3, \end{aligned}$$

where $\mathbf{x}_i, \mathbf{c}_i \in \mathbb{R}^{n_i}$ for $i = 1, 2$, $\mathbf{b}_j \in \mathbb{R}^{m_j}$ for $j = 1, 2, 3$, and each $A^{(i,j)}$ is an $m_j \times n_i$ real-valued matrix.

Exercise 2. For each of the four cases of the duality theorem, give (with justification) a corresponding example of a primal-dual pair of linear programs.

Exercise 3. Consider the following linear program P.

$$\begin{aligned} \min 4x_1 + 100x_2 + 120x_3 \text{ subject to} \\ x_i \geq 0 \text{ for } i = 1, 2, 3 \\ 15x_1 - x_2 - 3x_3 \leq 10 \\ -x_1 - x_2 - x_3 \leq -1 \end{aligned}$$

Find the dual program, and use it to find the optimal value of P.

[1] Just kidding, I don't have a fireplace, but I'll probably burn the late submissions anyway.

Exercise 4. We saw three formulations of the Farkas lemma, as given below.

> **Lemma 1.** *Let A by a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector.*
>
> *(i) The system $A\mathbf{x} = \mathbf{b}$ has a nonnegative solution if and only if every $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq 0$.*
>
> *(ii) The system $A\mathbf{x} \leq \mathbf{b}$ has a nonnegative solution if and only if every nonnegative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq 0$.*
>
> *(iii) The system $A\mathbf{x} \leq \mathbf{b}$ has a solution if and only if every nonnegative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A = \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq 0$.*

We showed (i) implies (ii) and (iii) implies (ii). Prove enough additional implications to show that (i), (ii) and (iii) are equivalent.

Exercise 5. Alice and Bob go out on a first date at a fancy, but not exorbitantly priced, restaurant, having been set up by their mutual friend Carol[2]. Having finished discussing the unseasonably warm winter shortly after their order was taken[3], the silence as they wait for their appetisers quickly becomes uncomfortable. Bob, an inveterate gambler, decides to break the tension with a zero-sum two-player game.

"Say now, Alice," he says, with a gleam in his eye, "would you fancy playing a game? We each pick a number, either one, two or three, and write it down on these here napkins. Once we've both chosen our numbers, I'll take them and add them together[4]. If the sum of the numbers is even, then I'll pay you \$10. If the sum is odd, you pay me \$10." After a moment's thought, he adds, "You do know what odd and even numbers are, right?"

Alice is slightly taken aback, for this was not the kind of proposition she was expecting to face on a first date, and somewhat infuriated by his offensive question. She is tempted to walk out, but the thought of the sumptious steak she ordered keeps her in her seat. Instead, she decides, she might as well teach Bob a lesson by winning his money. "Why yes," she replies, her voice cold as steel, "I do believe I am familiar with arithmetic modulo two. Let the games begin."

(i) Write down Alice's payoff matrix.

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[2]Carol shall play no further role in this story, but I have often been accused of neglecting to develop the characters of the supporting cast, so here is her résumé: having studied economics at Harvard (where Alice was her roommate in their junior and senior years) she quickly found employment with a financial consultancy firm. This job paid her well, but she truly loved it because of the frequent travel opportunities it presented her. While flying back from a business meeting in Japan, she had sat across the aisle from Bob, who struck her as a suitable match for Alice, hence this date.

[3]Their order was taken by Pedro, a young waiter still finding his feet in the service industry. While he did not particularly enjoy waiting tables, and found the starched shirts he was forced to wear rather uncomfortable, he endured these displeasures willingly, for he one day hoped to be promoted to the position of maître d'hôtel, a job he found much more appealing (and financially rewarding).

[4]His offer to do the adding was motivated not by chivalry but rather a desire to impress Alice with his quick arithmetic.

(ii) Identify all the (mixed) Nash equilibria. What is the value of the game?

With her background in economics, Alice plays according to an optimal strategy. However, after a few rounds, she quickly realises Bob is choosing his numbers uniformly at random.

(iii) How should Alice adjust her strategy to take advantage of Bob's ignorance?

With her new strategy, Alice starts winning regularly. Bob becomes alarmed – this date is quickly becoming more expensive than he had hoped. "Hey now," he whines, "this ain't fair! There are five ways you can win, but only four ways I win. I should only have to pay $8 when I lose[5]."

(iv) What is the value of the game and the Nash equilibria with these new payoffs? Should Alice agree to the new terms?

[5] For all his shortcomings in game theory, he remains a skilled calculator.