Algorithmic Combinatorics WS 2014 / 2015
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Exercise Sheet 12

Due date: Jan 30th, 2:00 PM, tutor box of Shagnik Das
Late submissions will be turned into paper aeroplanes and flown far, far away.

You should try to solve and write up all the exercises. You are welcome to submit **at most** three neatly written exercises for correction each week. You are encouraged to submit in pairs, but please indicate the author of each solution. Each problem is worth 10 points.

Exercise 1. A point $\vec{x}$ is called an *extreme point* of a convex set $C \subset \mathbb{R}^n$ if $\vec{x} \in C$ and there are no $\vec{y}, \vec{z} \in C$, both different from $\vec{x}$, such that $\vec{x}$ is on the line segment between $\vec{y}$ and $\vec{z}$.

(i) Show that the set $P \subset \mathbb{R}^n$ of feasible solutions to an n-variable linear program is a convex set.

(ii) Show that every basic feasible solution is an extreme point of P.

Exercise 2. Consider the following payoff matrix for a non-zero-sum two-player game, where Alice can choose a strategy from $\{A_1, A_2, A_3\}$, Bob can choose from $\{B_1, B_2, B_3\}$, and P_A (P_B) denotes the payoff to Alice (Bob).

(P_A, P_B)	B_1	B_2	B_3
A_1	$(4,7)$	$(6,-1)$	$(1,5)$
A_2	$(6,1)$	$(4,5)$	$(-3,0)$
A_3	$(2,3)$	$(5,5)$	$(4,1)$

(i) Show that neither A_3 nor B_3 would be played in any Nash equilibrium.

(ii) Show that there is no pure Nash equilibrium.

(iii) Find a mixed Nash equilibrium.

[Hint (to be read backwards): eugrA taht ni a dexim muirbiliuqe, eht detcepxe ffoyap ot boB nehw boB syalp ${}_1B$ dluohs lauqe taht nehw boB syalp ${}_2B$.]

Exercise 3. If $\vec{G}$ is a directed graph with n vertices and m arcs, we define a matrix $A(\vec{G})$ as follows. $A(\vec{G})$ is an $n \times m$ matrix with rows corresponding to the vertices of $\vec{G}$, and columns corresponding to the arcs of $\vec{G}$. The entry $A(\vec{G})_{v,e}$ on row $v \in V(\vec{G})$ and column $e \in E(\vec{G})$ is 1 if the arc e starts at v, -1 if the arc e ends at v, and 0 otherwise.

(i) Suppose $\vec{C}$ is a directed graph whose underlying undirected graph is a cycle ($\vec{C}$ itself need not be a directed cycle). Prove that $\det(A(\vec{C})) = 0$.

(ii) Prove that for any directed graph $\vec{G}$, $A(\vec{G})$ is totally unimodular.

(iii) Deduce the integrality theorem for the maximum flow problem.

Exercise 4. Let $\mathcal{D}$ be a finite collection of congruent discs in the plane, such that any two have a point in common. Show there exist 5 points in the plane such that every disc in $\mathcal{D}$ contains at least one of the points.