Algorithmic Combinatorics — WS 2014 / 2015

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Exercise Sheet 14

Due date: Feb 13th, 2:00 PM, tutor box of Shagnik Das
Late submitters may not[1] receive any feedback.

You should try to solve and write up all the exercises. You are welcome to submit your solutions to receive feedback. You are encouraged to submit in pairs, but please indicate the author of each solution.

Exercise 1. Suppose one can compute the determinant of an $n \times n$ matrix with $O(n^{\omega})$ arithmetic operations. There is a probabilistic algorithm that tests for the *existence* of perfect matchings in an n-vertex graph with $O(n^{\omega})$ operations.

Using this algorithm, develop an algorithm for *finding* a perfect matching in an n-vertex graph with m edges that requires $O(mn^{\omega} \log n)$ operations.

Exercise 2. The goal of this exercise is to prove $m(k) = O(k^2 2^k)$ via a randomised construction. Fix the ground set of elements $[2n]$, and consider the random hypergraph $\mathcal{F}$ with m edges, $F_1, F_2, \ldots, F_m$, chosen independently and uniformly at random (with repetition) from the family $\binom{[2n]}{k}$ of all k-sets in $[2n]$.

(i) Let χ be a (fixed) red/blue colouring of the elements $[2n]$. Show that, for each i, the probability of F_i being monochromatic is at least $2\binom{n}{k}/\binom{2n}{k}$.

(ii) If p is the lower bound from (i), show that $p = 2\prod_{j=0}^{k-1} \frac{n-j}{2n-j} \geq \left(\frac{n-k}{2n-k}\right)^k$. Given $1 - x \leq e^{-x} \leq 1 - \frac{1}{2}x$ for x sufficiently small, show that $p \geq 2^{-k}e^{-2k^2/(2n-k)}$ if n is sufficiently large with respect to k.

(iii) Deduce that the probability of there being a proper colouring of $\mathcal{F}$ is at most $2^{2n}(1-p)^m$.

(iv) By choosing appropriate values for m and n in terms of k, show that there exists a non-two-colourable k-graph $\mathcal{F}$ with $O(k^2 2^k)$ sets.

Exercise 3. Improve either[2] the upper or lower bound on $m(k)$ to $k2^k$.[3]

[1] Then again, they might - may the odds be ever in your favour.
[2] Bonus points for doing both!
[3] A successful solution might constitute a good start to a Ph.D.

Exercise 4. Show that the bound in the Erdős–Selfridge theorem is tight, i.e. $\tilde{m}(k) = 2^{k-1}$.

[Hint: redisnoC a yranib eert fo htped $k-1$, dna dliub a elbatius hpargrepyh. esU eht yrtemmys fo eht eert ot dnif a ygetarts rof M.]

Exercise 5. Suppose M and B play a more symmetric game on a k-graph $\mathcal{F}$. In every turn, first M colours one element red, and then B colours one element blue. The first player to have coloured (with his or her own colour) every element in some $F \in \mathcal{F}$ wins the game. If all the elements get coloured without either player managing to win, the game is a draw.

Show that for any k-graph, M has a strategy that guarantees her either a win or a draw.

Exercise 6. Let G be a bipartite graph with n vertices. Show that if every vertex of G is assigned a list of $\log_2(n)$ colours, G has a proper list-colouring.