Algorithmic Combinatorics WS 2014 / 2015
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Exercise Sheet 15

This sheet is designed to provide you with an opportunity to practise your mastery of the Local Lemma, the topic of the last week of lectures, as it will be examinable for the second exam.

Exercise 1. Extend the proof of the Local Lemma from class (for two-colouring hypergraphs) to the following more general theorem (which has the optimal constants).

Theorem 1 (Lovász Local Lemma)**.** *Let $E_1, E_2, \ldots, E_m$ be events in some probability space. Let $d \in \mathbb{N}$ and $p \in [0,1]$ be such that, for every $i \in [m]$, we have*

(1) $\mathbb{P}(E_i) \leq p$, and

(2) there is a set $\Gamma(i) \subseteq [m]\{i\}$ of at most d indices, such that the event E_i is mutually independent of $\{E_j : j \in [m] \setminus (\Gamma(i) \cup \{i\})\}$.

If $ep(d+1) \leq 1$, then with positive probability none of the events E_i occur.

It may help to show that for any $i \in [m]$ and $J \subseteq [m] \setminus \{i\}$, we have $\mathbb{P}\left(E_i | \cap_{j \in J} E_j^c\right) \leq ep$. You may use the estimate $(1 - 1/(d+1))^d \geq e^{-1}$.

Exercise 2. In class we showed that, for a k-uniform hypergraph $\mathcal{F}$ with $\Delta(L(\mathcal{F})) \leq 2^{k-4}$, the expected number of recolourings in the algorithmic Local Lemma is $O(m \log m)$. Show that, with more careful analysis, this bound can be greatly improved to $O\left(\frac{n}{k} \log m\right)$.

Exercise 3. Recall that the Ramsey number $R(k,k)$ is the smallest n such that any two-colouring of the edges of K_n must contain a monochromatic copy of K_k.

(i) By colouring edges randomly, show that if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. Deduce that $R(k,k) \geq \frac{1}{e\sqrt{2}}(1+o(1))k2^{k/2}$. [This is from Discrete Math I.]

(ii) Obtain a $\sqrt{2}$-factor improvement of the result in (i) by 'correcting' a random colouring by removing monochromatic cliques: show that for any integer n, $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$. Deduce that $R(k,k) \geq \frac{1}{e}(1+o(1))\, k2^{k/2}$.

(iii) Improve the bound by yet another $\sqrt{2}$-factor with the Local Lemma: show that if $e\binom{k}{2}\binom{n-2}{k-2} 2^{1-\binom{k}{2}} \leq 1$, then $R(k,k) > n$. Deduce the bound $R(k,k) \geq \frac{\sqrt{2}}{e}(1+o(1))k2^{k/2}$.

Exercise 4. The city of London is surrounded by the M25 motorway, a circular road that directs traffic around the city without congested its inner roads. It is approximately 110 miles long and, as per UK traffic regulations, has 30 streetlights per mile, and thus a total of 3300 lampposts.

To comply with recent environmental guidelines, the Mayor of London wants to illuminate the M25 with environmentally-friendly lightbulbs that will consume less power while maintaining adequate light coverage. To find the best lightbulb for the job, he commissions London's 300 different lighting firms to submit prototypes for evaluation.

Each firm provides a sample of 11 lightbulbs. To ensure that no firm has all its lightbulbs in a favourable stretch of the highway, all 3300 lightbulbs are mixed together and then placed, in some arbitrary order, in the M25's lampposts. The Mayor intends to keep these lightbulbs in place for a month and evaluate their efficiency before making a final decision about which lightbulb to use in the long-term.

Unfortunately, after a few days, he realises that this experiment is rather expensive, and decides the test has to be scaled down[1]. Thus one of each company's 11 lightbulbs will be switched off. However, in the interests of public safety, no two neighbouring lightbulbs should both be switched off, for fear of creating too long a dark stretch on the motorway.

Show that, regardless of how the lightbulbs were initially distributed, it is always possible to safely turn off one lightbulb from each company.

[1] An alternative would have been to raise taxes to fund the project, but he is a proud patriot, and, after a rather poor showing at the FIFA World Cup 2014, decides his country can ill afford to surrender either of her two advantages over France: lower taxes and finer food.