Algorithmic Combinatorics WS 2014 / 2015
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Exercise Sheet 5

Due date: Nov 21st, 2:00 PM, tutor box of Shagnik Das
Late submissions will be destroyed with extreme prejudice!

You should try to solve and write up all the exercises. You are welcome to submit **at most** three neatly written exercises for correction each week. You are encouraged to submit in pairs, but please indicate the author of each solution. Each problem is worth 10 points.

Exercise 1. We consider here a generalisation of the minimum spanning tree problem in terms of more abstract structures called *matroids*. A matroid is a pair $(X, \mathcal{B})$ of a finite ground set X and a collection $\mathcal{B}$ of subsets of X called *bases* that satisfy the following axioms:

(A1) There is at least one basis, i.e. $\mathcal{B} \neq \emptyset$.

(A2) The basis exchange property: If $A, B \in \mathcal{B}$ with $A \neq B$, then for every $a \in A \setminus B$ there is some $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$ is another basis.

A set A is called *independent* if it is a subset of some basis; that is, if there is $B \in \mathcal{B}$ with $A \subseteq B$.

(i) Show that all bases in a matroid must have the same cardinality.

(ii) Show that if $G = (V, E)$ is a connected graph, and $\mathcal{T}$ is the set of spanning trees of G, then $(E, \mathcal{T})$ forms a matroid. What are the independent sets in this matroid?

If we assign nonnegative weights $w : X \to \mathbb{R}_{\geq 0}$ to elements in the ground set, we can then ask for the minimum weight basis; that is, for $B \in \mathcal{B}$ minimising $\sum_{x \in B} w(x)$.

The greedy algorithm is as follows: let X be ordered by weight, so $X = \{x_1, x_2, \ldots, x_n\}$ with $w(x_1) \leq w(x_2) \leq \ldots \leq w(x_n)$. Start with $S_0 = \emptyset$. At time t, for $1 \leq t \leq n$, let $T = S_{t-1} \cup \{x_t\}$. If T is independent, set $S_t = T$, and otherwise set $S_t = S_{t-1}$. The output of the algorithm is the final set S_n.

(iii) Show that the greedy algorithm produces a basis of minimum weight.

Bonus (10 pts): Show that if $\mathcal{F}$ is a collection of subsets of a finite ground set X such that for any nonnegative weight function $w : X \to \mathbb{R}_{\geq 0}$, the greedy algorithm always produces a set $F \in \mathcal{F}$ of minimal weight, then $(X, \mathcal{F})$ is a matroid.

Exercise 2. Given a graph $G = (V, E)$, one could try to apply the greedy algorithm to find a maximum matching of G. Order the edges $E = \{e_1, e_2, \ldots, e_m\}$ in some (arbitrary) way. Start with $M_0 = \emptyset$. At time t, for $1 \leq t \leq m$, if $M_{t-1} \cup \{e_t\}$ is a matching, set $M_t = M_{t-1} \cup \{e_t\}$. Otherwise set $M_t = M_{t-1}$. Return the final matching M_m.

Show this gives a $\frac{1}{2}$-approximation algorithm for the maximum matching of G.

Exercise 3. Suppose we have K_n, the complete graph on n vertices, and a weighting of its edges $w : E(K_n) \to [1, \infty)$ satisfying the following weakened triangle inequality for all triples $x, y, z \in V(K_n)$:

$$w(\{x, y\}) \leq w(\{x, z\}) + w(\{z, y\}) + 10.$$

Find a 12-approximation algorithm for the Travelling Salesman Problem on this weighted graph.

Exercise 4. An *n-permutation* matrix is an $n \times n$ matrix with $\{0, 1\}$ entries such that every row and every column contains precisely one 1. Show that an $n \times n$ matrix with nonnegative integer entries can be written as the sum of k n-permutation matrices if and only if all row and column sums are equal to k.

Exercise 5. Show that every graph $G = (V, E)$ contains a subset of the vertices $S \subset V$ such that $2\alpha'(G) = |V| + |S| - o(G - S)$.

Exercise 6. Using the result from Exercise 5, show that if G is a simple graph with minimum degree $\delta(G) \geq k$ and $|V(G)| \geq 2k$, then $\alpha'(G) \geq k$.