Algorithmic Combinatorics WS 2014 / 2015
Tibor Szabó
Shagnik Das

Exercise Sheet 6

Due date: Nov 28th, 2:00 PM, tutor box of Shagnik Das
Late submissions will be exterminated!

You should try to solve and write up all the exercises. You are welcome to submit **at most** three neatly written exercises for correction each week. You are encouraged to submit in pairs, but please indicate the author of each solution. Each problem is worth 10 points.

Exercise 1. Use the Hungarian Algorithm to find a minimum-weight matching in $K_{5,5}$ with the edge weights $W = (w_{i,j})$ as below, and then give a short proof that your matching is optimal. (It should go without saying, but you must show your work clearly.)

$$W = \begin{pmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{pmatrix}.$$

Exercise 2. Imagine, if you will, a kingdom (that looks very much like England) comprised of $2t$ provinces. Within this kingdom, a civil war is raging between the Lannisters, who control t of the provinces, and the Northerners, who control the other t. The ith Lannister province has a population of ℓ_i, while the jth Northern province has a population of n_j. One day they decide to go to war, and so their leaders go to speak to the King to ask how they may fight.

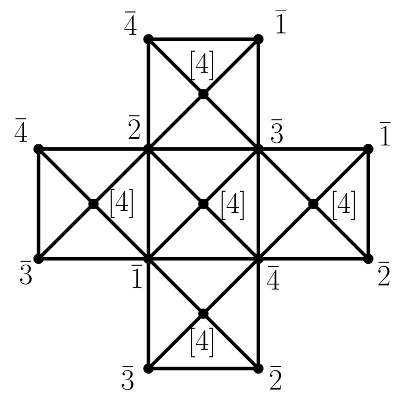
The king, who runs a chessboard-making business on the side, decides that the only fair way to wage a civil war is through games of chess. He pairs up the provinces, so that each province will fight against *one* province from the other side. The 'fights' between a pair of provinces consist of *every* person from the Lannister province playing a game of chess against *every* person from the corresponding Northern province. Whichever side wins the majority of the chess games wins the civil war.

To maximise his profits, the King seeks to maximise the total number of chess games played. How should he pair up the provinces?

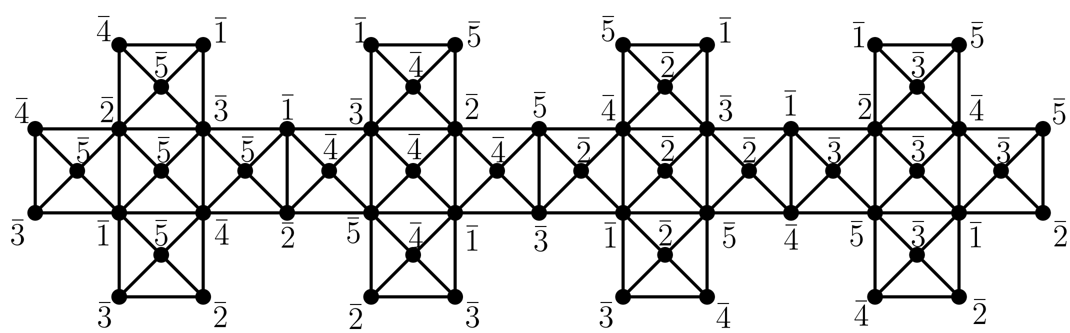
Exercise 3. Determine $\chi_\ell(K_{2,4})$.

Exercise 4. We have seen that all planar graphs are 5-choosable. This example shows that you cannot do better – there are planar graphs that are not 4-choosable.

(i) Show that the following graph has no proper coloring from the assigned lists. (In this part of the exercise, $\bar{i}$ denotes the set $[4] \setminus \{i\}$.)



(ii) Let G be the graph obtained from the graph below by adding an extra vertex to the outside face and connecting it to all the vertices on the boundary. Show that if the new vertex is assigned the list $\bar{1}$, then there is no proper list coloring of G from the assigned lists. (In this part of the exercise, $\bar{i}$ denotes the set $[5] \setminus \{i\}$.)



Exercise 5. Let D be a directed graph, with each vertex v having at most d in-edges and at most d out-edges. Show that the edges of D can be coloured with d colours so that any two edges with the same start-vertex or the same end-vertex receive different colours.