Algorithmic Combinatorics — WS 2014 / 2015

Tibor Szabó
Shagnik Das

Exercise Sheet 7

Due date: Dec 5th, 2:00 PM, tutor box of Shagnik Das
Late submissions will be frowned upon most sternly (and discarded thereafter)!

You should try to solve and write up all the exercises. You are welcome to submit **at most** three neatly written exercises for correction each week. You are encouraged to submit in pairs, but please indicate the author of each solution. Each problem is worth 10 points.

Exercise 1. Formulate and prove a mathematically precise statement expressing the following: the stable matching produced by the proposal algorithm with the men proposing is the best possible stable matching for the men.

Exercise 2. Let G be a graph with maximum degree $\Delta = \Delta(G)$.

(i) Show that if the vertices of G of degree Δ induce a forest, then G is Δ-edge-colourable.

(ii) Show that for every matching M in G, there is a $(\Delta+1)$-edge-colouring of G in which all edges of M receives the same colour.

Bonus (5 pts): Show that for any matching M in G, any assignment of colours from $[\Delta+1]$ to the edges of M can be extended to a proper $(\Delta+1)$-edge-colouring of G.

Exercise 3. Let G be a bipartite graph. An *orientation* of G is a directed graph $\vec{D}$ where every edge $\{u,v\}$ of G is assigned *one* of the orientations (u,v) or (v,u) in $\vec{D}$.

(i) Let $\vec{D}$ be an orientation of G where every vertex has an out-neighbour. Show that $\vec{D}$ has a kernel.

(ii) Show that every orientation $\vec{D}$ of G has a kernel.

(iii) Deduce that every bipartite digraph is kernel-perfect.

Bonus (5 pts): Show that every bipartite graph with maximum degree Δ is $\left(\left\lceil \frac{\Delta}{2} \right\rceil + 1\right)$-list-colourable.

Exercise 4. Your mission, should you choose to accept it, is to extend the proof of Galvin's theorem from $K_{n,n}$ to general bipartite graphs.

(i) Considering the complete bipartite graph $G = K_{n,n}$ with vertices $X \cup Y$, show that the edge-colouring $\phi(\{x_i, y_j\}) = i + j \mod n$ gives a proper n-edge-colouring of G.

(ii) Let G now be an arbitrary bipartite graph with edge-chromatic number $\chi'(G) = n$. Use an n-colouring of $E(G)$ to find an orientation $\vec{D}$ of the line graph $L(G)$ such that:

   (a) The maximum out-degree is at most $n - 1$.

   (b) Given an edge $\{x, y\}$ of G, the digraphs induced on the sets $S_x = \{\{x, y'\} : y' \in N(x)\} \subset V(L(G))$ and $S_y = \{\{x', y\} : x' \in N(y)\} \subset V(L(G))$ are transitive.

(ii) Show that $\vec{D}$ is kernel-perfect, and deduce that $\chi'_\ell(G) = n$.

Exercise 5. Let G be a connected graph with an even number of edges. By considering the line graph of G, show that the edges of G can be partitioned into paths of length two.

Exercise 6. There are n women, $W = \{w_i : 1 \leq i \leq n\}$, and n men, $M = \{m_i : 1 \leq i \leq n\}$, and Cupid would like to find a stable matching between the two sets. Cupid starts by asking each person their preferences. Each woman w_i has permutation $\pi_i \in S_n$, where $\pi_i(j) = k$ means that w_i finds m_j to be the kth most attractive man. Each man m_i has a similar permutation $\sigma_i \in S_n$ for his preferences among the women.

Unfortunately, Cupid did not pay attention in class, and cannot remember the algorithm for finding a stable matching. However, Cupid does remember the Hungarian algorithm for finding a maximum-weight matching. Reasoning that in a stable matching, both partners should find each other highly attractive, Cupid forms a weighted bipartite graph on $W \cup M$ with edge weights $\omega(w_i, m_j) = (n - \pi_i(j)) + (n - \sigma_j(i))$.

Show that there can be permutations $\{\pi_i : i \in [n]\}$ and $\{\sigma_i : i \in [n]\}$ such that no maximum-weight matching is a stable matching.

Bonus (5 pts): Given the preference permutations, is it always possible to find strictly monotone decreasing functions $\{f_i : i \in [n]\}$ and $\{g_i : i \in [n]\}$ such that a maximum matching with respect to the weights $\omega(w_i, m_j) = f_i(\pi_i(j)) + g_j(\sigma_j(i))$ is a stable matching? Can we always take all the functions f_i and g_j to be the same function f?