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Exercise Sheet 8

Due date: Dec 12th, 2:00 PM, tutor box of Shagnik Das
Late submissions will mysteriously disappear.

You should try to solve and write up all the exercises. You are welcome to submit **at most** three neatly written exercises for correction each week. You are encouraged to submit in pairs, but *please indicate the author of each solution.* Each problem is worth 10 points.

Exercise 1. The city of Berlin is populated with people of diverse backgrounds, but they share one feature: an appreciation for the finer things in life. As such, all the citizens of Berlin (at least, all the citizens that this problem is concerned with) are fans of Cristiano Ronaldo. Being fans, they would like to travel from Berlin to Zürich to see him receive his Ballon d'Or award on the 12th of January. Being a wealthy, benevolent and kind-spirited superstar, Cristiano offers to pay the travel costs for any of his fans from Berlin who will come to Zürich for the ceremony. Naturally, this creates a great deal of demand for Berlin to Zürich transportation (most want to see Cristiano, but some want chocolate).

The poor Deutsche Bahn service is overwhelmed with ticket requests, and they turn to you for help with planning people's journeys. They show you the list of train routes they have available, as well as the number of free seats remaining. 'Ber' denotes Berlin, and 'Zür' denotes Zürich.

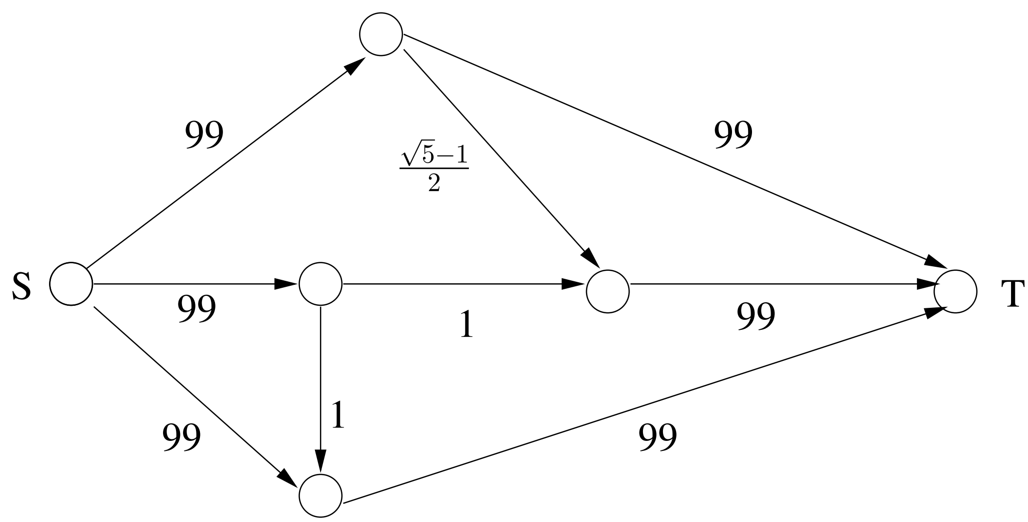
Origin	→	Destination	Seats	Origin	→	Destination	Seats
Ber	→	Dre	500	Ber	→	Ham	1400
Dre	→	Pra	400	Dre	→	Mün	400
Dre	→	Sch	500	Pra	→	Mün	300
Ham	→	Dre	1300	Ham	→	Fra	200
Ham	→	Boc	500	Sch	→	Ham	600
Sch	→	Fra	600	Mün	→	Sch	400
Mün	→	Fra	300	Mün	→	Zür	600
Boc	→	Fra	300	Fra	→	Zür	1200

Given this information, find a routing plan that will carry the maximum possible number of Ronaldo fans from Berlin to Zürich. Remember that the fans who you do not have space for will be very upset, so you should have a very convincing (and preferably short - upset people are not always patient) proof that your plan is the best possible.

Bonus (5 pts): Jealous of all of Ronaldo's fans, Messi decides to intervene. He plans on inciting a strike on the Deutsche Bahn, so that Ronaldo's fans will not be able to travel to support him in Zürich. However, due to Messi's waning influence, he only has enough leverage to make one of the train lines stop running. Which line should Messi stop to minimise the number of fans who can travel to Zürich?

Bonus bonus (0 pts[1]): Can you fill in the names of the other cities in this travel network?

Exercise 2. Consider the network in the figure below. The source and the sink are marked with S and T, and the capacity of every edge is indicated.



(i) Find (with proof) the value of the maximum flow in the network.

(ii) Describe a choice of augmenting paths in the Ford-Fulkerson algorithm for which the algorithm never finishes and the flow value converges to $2 + \sqrt{5}$.

[Hint (reverse all words): ruoY mhtirogla dluohs esu eerht gnitnemgua shtap, htiw egde seiticapac a-c-b-b-a, a-b-c-a, dna a-b-b-a ylevitcepser, erehw a, b, c era 99, 1, $\frac{\sqrt{5}-1}{2}$ ylevitcepser.]

[1] Not a typo - my frivolity has limits!

Exercise 3. H is a maximal k-connected subgraph of G if:

1. H is a subgraph of G.

2. H is k-connected.

3. If $H \subsetneq H' \subset G$, then H' is not k-connected.

Show $|V(H_1) \cap V(H_2)| \leq k - 1$ for two distinct maximal k-connected subgraphs H_1 and H_2.

Exercise 4. Prove that every k-regular graph G with $2n$ vertices and $\kappa'(G) \geq k - 1$ has a perfect matching.

Exercise 5. Suppose $G = (V, E)$ is a connected graph. Show that $\kappa'(G) \geq k$ if and only if for every $xy \in E(G)$, there are cycles $C_1, C_2, \ldots, C_{k-1} \subset E(G)$ such that for all $i \neq j$, $C_i \cap C_j = \{xy\}$.