Algorithmic Combinatorics WS 2014 / 2015
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Exercise Sheet 9

Due date: Dec 19th, 2:00 PM, tutor box of Shagnik Das
Late submissions will be used as confetti for our Christmas party.

You should try to solve and write up all the exercises. You are welcome to submit **at most** three neatly written exercises for correction each week. You are encouraged to submit in pairs, but please indicate the author of each solution. Each problem is worth 10 points.

Exercise 1. Five teams compete in a children's football league, where a win is worth 2 points, a draw worth 1 point, and a loss worth 0 points. With the season partially complete, the league standings are given below, with the number of remaining games against each team also given.

							Games	remaining	against	
Team	Pld	W	D	L	Pts	Hertha	Union	Schalke	Dortmund	Bremen
Hertha	22	6	9	7	21	–	0	1	0	1
Union	12	8	2	2	18	0	–	2	7	3
Schalke	19	6	2	11	14	1	2	–	0	2
Dortmund	12	3	7	2	13	0	7	0	–	5
Bremen	13	4	4	5	12	1	3	2	5	–

At the end of the season, a team is the sole champion if they have strictly more points than any of the other teams. Use network flows to determine which of Hertha and Dortmund can go on to become sole champions. Justify your solution fully.

Bonus (10 pts): What is the highest possible ranking that Schalke can achieve, and what sequence of results is required for such an outcome?

Exercise 2. Given a weighted graph $G = (V, E)$, we say the *value* of a spanning tree is the minimum weight of its edges. We say the *capacity* of a cut $[S, V \setminus S]$ is the maximum weight of the edges in the cut. Show that the maximum value of a spanning tree in G is equal to the minimum capacity of a cut.

Exercise 3. Suppose $n \geq 2$. Baranyai's Theorem guarantees $\binom{[3n]}{3}$ can be partitioned into perfect matchings without explicitly describing these matchings. In this exercise you will give such an explicit description in the case when $p = 3n - 1$ is a prime number.

(i) Consider the field $\mathbb{F}_p$, and denote by $\mathbb{F}_p^*$ the set of invertible elements, namely $\mathbb{F}_p^* = \{1, 2, \ldots, p-1\}$. Define the map $\pi : \mathbb{F}_p^* \to \mathbb{F}_p$ by $\pi(x) = -(1+x)x^{-1}$. Show that π is injective and $\pi^3(x) = x$ for any $x \neq p - 1$.

(ii) Add a new element u to $\mathbb{F}_p$, and extend π to $\{u, 0\}$ injectively so that $\pi^3(x) = x$ for all $x \in \mathbb{F}_p \cup \{u\}$. Show that this gives some perfect matching M_0 in $\binom{[3n]}{3}$.

(iii) By considering affine transformations $x \mapsto ax + b$, find another $\binom{3n-1}{2} - 1$ perfect matchings in $\binom{[3n]}{3}$.

(iv) Show that these matchings partition $\binom{[3n]}{3}$ into perfect matchings.

Exercise 4. A mysterious disease called Greyscale is sweeping through the city of King's Landing, population n. The nation's best epidemiologists study how the disease could spread through the population, and find that there is an orientation $\vec{T}$ of the complete graph on the population $V = [n]$ describing its progression: if a person $v \in V$ is infected, then in the evening she will infect all of her out-neighbours $u \in N^+(v)$.

(i) Show there is some set $S \subset V$ of $\lceil \log_2 n \rceil$ people such that if everyone in S is infected in the morning, then everyone in V will be infected by the end of the day.

(ii) Show there is one person $v^* \in V$ who, if infected, will cause everyone in V to be infected within two days.

(iii) Unfortunately, v^* is indeed infected, and by the end of the day has infected all of his out-neighbours. The next morning, doctors find a treatment that slows the spread of the disease, so that infected people can now only infect at most two of their out-neighbours (although there is no control over which two neighbours might get infected). Show that, no matter what the orientation $\vec{T}$ is, it is still possible for the entire population V to be infected by the end of this second day.

Bonus (10 pts): Show that the $\lceil \log_2 n \rceil$ bound in (i) is asymptotically tight; that is, for any $\varepsilon > 0$ and sufficiently large n, there is an orientation $\vec{T}$ of K_n such that no set of $(1-\varepsilon)\log_2 n$ vertices can infect all other vertices within a single day.

Exercise 5. A prominent toy company, who have requested to remain unnamed, are about to launch a new product ahead of this year's Christmas sales. They will place a minimum of 10 advertisements to promote the product, but could place more.

As per company policy, to avoid wasting supplies, the number of units of the toy they manufacture will be limited by how much it is promoted. If the bare minimum of 10 advertisements are used, they will not bother producing any units. However, for each additional advertisement, they will produce 50 units of the toy.

Their market research indicates that there is a current demand (without any advertisements) for 1000 units. They anticipate that each advertisement will change the demand (additively) by λ. If the advertisements are well received, they will increase the demand, while if the advertisements are bad, then the negative publicity will cause the demand to drop.

The company makes a profit of \$200 on each unit sold, but every advertisement costs them \$10. In the spirit of Christmas, the company seeks to maximise their total profits (after deducting advertising costs).

(i) Formulate an appropriate linear programming problem (where, for simplicity, fractional values of units and ads are allowed).

(ii) For each value of $\lambda \in \mathbb{R}$, determine if the linear program is infeasible, unbounded, or has an optimum. In the latter case, find the optimum and all optimal solutions.