4.2.2015 Alg. Comb: Lesson Plan

I. Review

A. Framework

1. M - red, B - blue, M goes first & they alternate.
2. B goal: colour one element from every $F \in \mathcal{F}$.
3. M goal: colour every element from one $F \in \mathcal{F}$.

~~B. Theorem~~

~~1. Thm: (Erdős-Selfridge)~~

B. Extremal definition

1. Def^n: $\tilde{m}(k) = \min \{ |\mathcal{F}| : B \text{ has no winning strategy for } \mathcal{F} \}$

C. Theorem

1. Thm 1: (Erdős-Selfridge)
$$\tilde{m}(k) \geq 2^{k-1}.$$

II. Erdős-Selfridge Proof

A. Non-uniform statement

1. We start with a k-graph
2. As game progresses, elements get coloured → lose uniformity
3. Thm 1 follows from following more general statement.
4. Thm 2: (Erdős-Selfridge, non-uniform version)
Let $\mathcal{F}$ be a collection of sets, with repetition, such that $\sum_{F \in \mathcal{F}} 2^{-|F|} < 1$.
Then, if M & B play on $\mathcal{F}$ with B going first, B has a winning strategy.

B. Deduction of uniform version

1. Pf of Thm 1 from Thm 2:
a. We have m sets of size k, $m < 2^{k-1}$, but M going ~~first~~.
b. After M's move, we have m sets of size $\geq k-1$ in $\mathcal{F}'$, with B going first.
c. $\sum_{F \in \mathcal{F}'} 2^{-|F|} \leq m 2^{-(k-1)} = \frac{m}{2^{k-1}} < 1.$

d. Thm 2 $\Rightarrow$ B has a winning strategy from here.

e. $\Rightarrow \tilde{m}(k) \geq 2^{k-1}$. $\square$

C. Proof of non-uniform theorem

1. Proof by induction on $\left|\bigcup_{F\in\mathcal{F}} F\right|$ $\leftarrow$ size of ground set.
2. Base case: $\left|\bigcup_{F\in\mathcal{F}} F\right| = 0$
   a. If $F \in \mathcal{F}$, then $F = \emptyset \Rightarrow 2^{-|F|} = 1$.
   b. $\sum_{F\in\mathcal{F}} 2^{-|F|} < 1 \Rightarrow \mathcal{F} = \emptyset$
   c. $\Rightarrow$ B wins (trivially has an elt from every $F \in \mathcal{F}$).
3. Induction step
   a. Define danger$(F) = 2^{-|F|}$, $F \in \mathcal{F}$.
   b. For $x \in \bigcup_{F\in\mathcal{F}} F$, let wt$(x) = \sum_{F \ni x}$ danger(x).
   c. ~~Let~~ B chooses x_1 of maximum weight
      (i) Break ties arbitrarily
   d. Let x_2 be M's subsequent play.
   e. After these two rounds, we are faced with a new game $\mathcal{F}'$, where
      $\mathcal{F}' = \{F - \{x_2\} : F \in \mathcal{F}, x_1 \notin \mathcal{F}\}$, with B to play first.
      (i) If $F \in \mathcal{F}$, $x_1 \in F$, then B has already claimed an element from F, so need not worry about it any more.
   f. Let danger$'(F)$ denote the danger of F in $\mathcal{F}'$.
      ~~(i)~~
      (i) $$\text{danger}'(F) = \begin{cases} 0 & \text{if } x_1 \in F \\ 2\,\text{danger}(F) & \text{if } x_1 \notin F,\ x_2 \in F \\ \text{danger}(F) & \text{if } x_1, x_2 \notin F. \end{cases}$$

~~g. $\Rightarrow \sum_{F\in\mathcal{F}'} 2^{-|F|} = \sum_{F\in\mathcal{F}} 2^{-|F|} - \sum_{x_1\in F} \text{danger}(F)$~~

g. $$\Rightarrow \sum_{F\in\mathcal{F}'} 2^{-|F|} = \sum_{F\in\mathcal{F}} \text{danger}'(F)$$
$$= \sum_{F\in\mathcal{F}} \text{danger}(F) - \sum_{F\ni x_1} \text{danger}(F) + \sum_{\substack{F\ni x_2 \\ F\not\ni x_1}} \text{danger}(F)$$
$$\leq \sum_{F\in\mathcal{F}} \text{danger}(F) - \text{wt}(x_1) + \text{wt}(x_2)$$
(x_1 of max weight) $$\leq \sum_{F\in\mathcal{F}} \text{danger}(F) < 1.$$

$(|\bigcup_{F\in\mathcal{F}'} F| \le |\bigcup_{F\in\mathcal{F}} F| - 1)$.

h. Induction $\Rightarrow$ B has a winning strategy for $\mathcal{F}'$. $\square$

D. Remarks

1. danger(F) = probability of F being all red if the remaining elts are coloured randomly
   a. $\therefore$ this proof is inspired by previous prob. proof.
2. B's strategy: <u>greedy</u>!
   a. At every round, choose the element of greatest weight
   b. $\Rightarrow$ total danger always < 1.
3. Then 1 is <u>best-possible</u>; c.f. homework.

III. <u>Improved lower bound, Take 1</u>

A. History

do this quickly, verbally!

1. Erdős proved $m(k) \ge 2^{k-1}$, conjectured $m(k)$ grows faster than 2^k.
2. Beck ('77, '78) proved $m(k) \ge c \log k \, 2^k$, then $m(k) \ge k^{1/3 + o(1)} 2^k$
3. Radhakrishnan–Srinivasan ('00) proved $m(k) \ge 0.7 \sqrt{\frac{k}{\ln k}} \cdot 2^k$
   a. Involved proof:
      (i) first colour randomly
      (ii) identify dangerous vxs, and (randomly) recolour.

B. Pluhár proof

1. In 2009, Pluhár gave a simplified (but weaker) improvement of the Erdős bound.
2. Instead of colouring randomly, as in previous proofs, Pluhár coloured <u>greedily</u>
   (i) Colour vertices red, unless that would make an all-red edge
   (ii) Process vertices in a <u>random order</u>.

C. Result.

1. <u>Thm</u> (Pluhár)

$$m(k) > c\, k^{\frac{1}{4}} 2^k \quad \text{for some constant } c>0.$$

2. <u>Pf</u>: a. Order the vertices uniformly at random

b. In this order, colour each vertex red, unless it is the last vx of an all-red edge, in which case we colour it blue.

c. Cannot create any all-red edges, only all-blue.

d. If F is an all-blue edge, consider the first vx in F.

(i) Coloured blue because it was the last vx in an all-red edge E.

(ii) say <u>E precedes F</u> if $|E \cap F| = 1$ and $E \leq F$ in the order (ptwise)

e. for given $E, F \in \mathcal{F}$,

$$\mathbb{P}(E \text{ precedes } F) = \frac{\overbrace{(k-1)!}^{\text{order } E\setminus F}\;\overbrace{(k-1)!}^{\text{order } F\setminus E}}{\underbrace{(2k-1)!}_{\text{order } E\cup F}}$$

f. $\therefore \mathbb{P}(\text{not proper colouring}) = \mathbb{P}(\exists \text{ blue edge})$

$$\leq \mathbb{P}(\exists\, E, F : E \text{ precedes } F)$$

$$\leq \sum_{E,F} \mathbb{P}(E \text{ precedes } F)$$

$$= m^2 \cdot \frac{(k-1)!\,(k-1)!}{(2k-1)!} \sim c^2 m^2 k^{-\frac{1}{2}} 2^{-2k}.$$

$\therefore$ If $m < c\, k^{\frac{1}{4}} 2^k$, $\mathbb{P}(\text{not proper}) < 1$

$\Longrightarrow \exists$ proper colouring.

$\Longrightarrow m(k) \geq c\, k^{\frac{1}{4}} 2^k$. $\square$

D. <u>Remark</u>

1. Very natural algorithm; ingenious use of randomness makes it work!

IV. <u>Improved lower bound, take two</u>

A. Improvement

1. In 2014, Cherkashin-Kozik modified the Pluhár result to match the best-known bound.

B. Statement

1. <u>Thm</u> (Cherkashin - Kozik)

$$m(k) \geq c \sqrt{\frac{k}{\ln k}}\, 2^k \text{ for some } c > 0.$$

C. Algorithm

1. For each vertex $x \in X$, assign a uniform random variable $U_x \sim U[0,1]$, independently.
2. Rank the vertices in increasing order of the U_x.
3. In this order, colour greedily:
   a. Red if no all-red edge
   b. Otherwise blue.

D. Remark

1. Exactly the same algorithm as before! (prompt students)
2. As before, can only have all-blue edges, and only if an all-red edge <u>precedes</u> the all-blue edge.

E. Analysis.

1. ~~Fix some~~ Let $p \in [0, \frac{1}{2})$ (to be fixed later).
2. Set $L = [0, \frac{1}{2} - p]$, $M = (\frac{1}{2} - p, \frac{1}{2} + p)$, $R = [\frac{1}{2} + p, 1]$.
3. Events:
   a. Call an edge $F \in \mathcal{F}$ a <u>left-edge</u> if $U_x \in L \ \forall x \in F$.
   b. Call an edge $F \in \mathcal{F}$ a right-edge if $U_x \in R \ \forall x \in F$
   c. Call a pair of edges middle-preceding if E precedes F and for $\{x\} = E \cap F$, $U_x \in M$.
4. Implications:

$$\{\text{not proper}\} = \{\exists \text{ all-blue edge}\} \subseteq \{\exists \text{ preceding pair}\}$$
$$\subseteq \{\exists \text{ left edge}\} \cup \{\exists \text{ right edge}\} \cup \{\exists \text{ middle preceding}\}$$

5. $\therefore \mathbb{P}(\text{not proper}) \leq \mathbb{P}(\exists \text{ left edge}) + \mathbb{P}(\exists \text{ right edge}) + \mathbb{P}(\exists \text{ mid. prec. pair})$

6. Extreme edges

a. $\mathbb{P}(\exists \text{ left edge}) \leq \underbrace{m}_{\#\text{ edges}} \cdot \underbrace{(\tfrac{1}{2}-p)^k}_{\text{prob. } U_x \leq \frac{1}{2}-p \ \forall x \in F}$

b. similarly, $\mathbb{P}(\exists \text{ right edge}) \leq m(\tfrac{1}{2}-p)^k$.

7. Middle preceding pair:

a. $\mathbb{P}(E, F \text{ middle preceding})$

$$= \int_{\frac{1}{2}-p}^{\frac{1}{2}+p} \underbrace{u^{k-1}}_{\text{prob of left elt. smaller}} \underbrace{(1-u)^{k-1}}_{\text{prob of right elts bigger}} \underbrace{du}_{\text{position of mid. elt.}}$$

$$\leq \int_{\frac{1}{2}-p}^{\frac{1}{2}+p} \left(\tfrac{1}{4}\right)^{k-1} du = (2p) \cdot 2^{-2(k-1)}.$$

b. $\Rightarrow \mathbb{P}(\exists E, F \text{ middle preceding}) \leq m^2 \cdot (2p) \cdot 2^{-2(k-1)}$.

8. Estimates

a. $\Rightarrow \mathbb{P}(\text{not proper}) \leq 2m(\tfrac{1}{2}-p)^k + m^2(2p)2^{-2(k-1)}$

$$\leq 2 \cdot m \cdot 2^{-k} \cdot e^{-2pk} + 8 \cdot m^2 \cdot p \cdot 2^{-2k}$$

b. Take $m = c\sqrt{\frac{k}{\ln k}}\, 2^k$, $p = \frac{1}{2}k^{-1} \ln\sqrt{\frac{k}{\ln k}}$

$\Rightarrow \mathbb{P}(\text{not proper}) \leq 2c \cdot \sqrt{\frac{k}{\ln k}}\, e^{-\ln\sqrt{\frac{k}{\ln k}}} + 4c^2 \cdot \frac{k}{\ln k}\, k^{-1} \ln\sqrt{\frac{k}{\ln k}}$

$\leq 2c + 2c^2 < 1$ for c small enough.

$\Rightarrow \exists$ proper colouring. $\square$.

Need to get through to here !!!