Recall: Leaves, trees, forests...

A graph with no cycle is acyclic. An acyclic graph is called a forest.

A connected acyclic graph is a tree.

A leaf (or pendant vertex) is a vertex of degree 1.

A spanning subgraph of G is a subgraph with vertex set $V(G)$.

A spanning tree is a spanning subgraph which is a tree.

Examples. Paths, stars

Recall: Properties of trees

Lemma. T is a tree, $n(T) \geq 2 \Rightarrow T$ contains at least two leaves.
Deleting a leaf from a tree produces a tree.

Theorem (Characterization of trees) For an n-vertex graph G, the following are equivalent

1. G is connected and has no cycles.
2. G is connected and has $n - 1$ edges.
3. G has $n - 1$ edges and no cycles.
4. For each $u, v \in V(G)$, G has exactly one u, v-path.

Corollary.

(i) Every edge of a tree is a cut-edge.

(ii) Adding one edge to a tree forms exactly one cycle.

(iii) Every connected graph contains a spanning tree.

Recall: The edge-exchange lemma

Proposition. Let T and T' be spanning trees of a connected graph G.
Then for every $e \in E(T) \setminus E(T')$, **there exists** an edge $e' \in E(T') \setminus E(T)$, such that both $T - e + e'$ and $T' + e - e'$ are spanning trees of G.

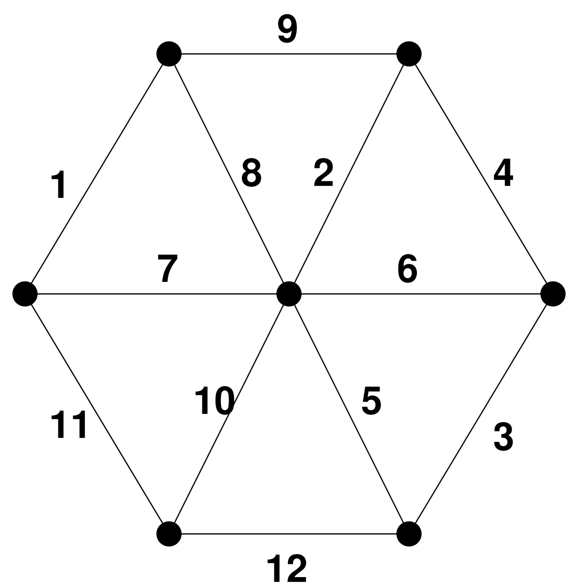
How to build the cheapest road network?

G is a weighted graph if there is a weight function $w : E(G) \to \mathbb{R}$.

Weight $w(H)$ of a subgraph $H \subseteq G$ is defined as

$$w(H) = \sum_{e \in E(H)} w(e).$$

Example:



Kruskal's Algorithm

Input: connected graph G, weight function $w : E(G) \to \mathbb{R}$, $w(e_1) \leq w(e_2) \leq ... \leq w(e_m)$.

Idea: Maintain a spanning forest H of G. At each iteration try to enlarge H by an edge of smallest weight.

Initialization: $V(H) \leftarrow V(G)$, $E(H) \leftarrow \emptyset$, $i \leftarrow 1$

WHILE $i \leq n$
    $e \leftarrow e_i$
    IF e goes between two components of H THEN
        update $H \leftarrow H + e$
        IF H is connected THEN
            stop and return H
    $i \leftarrow i + 1$

Theorem. In a connected weighted graph G, Kruskal's Algorithm constructs a minimum-weight spanning tree.

Proof of correctness of Kruskal's Algorithm

Proof. T is the graph produced by the Algorithm.
$E(T) = \{f_1, \ldots, f_{n-1}\}$ and $w(f_1) \le \cdots \le w(f_{n-1})$.

Easy: T is spanning (already at initialization!)
T is a connected (by termination rule) and has no cycle (by iteration rule) $\Rightarrow T$ is a tree.

But **WHY** is T min-weight?

Let T^* be an arbitrary min-weight spanning tree. Let j be the largest index such that $f_1, \ldots, f_j \in E(T^*)$.

If $j = n - 1$, then $T^* = T$. Done.

Proof of Kruskal, cont'd

If $j < n-1$, then $f_{j+1} \notin E(T^*)$.
There is an edge $e \in E(T^*)$, such that
$T^{**} = T^* - e + f_{j+1}$ is a spanning tree.

(i) $w(T^*) - w(e) + w(f_{j+1}) = w(T^{**}) \geq w(T^*)$
So $w(f_{j+1}) \geq w(e)$.

(ii) Key: When we selected f_{j+1} into T, e was also available. (The addition of e wouldn't have created a cycle, since $f_1, \ldots, f_j, e \in E(T^*)$.)
So $w(f_{j+1}) \leq w(e)$.

Combining: $w(e) = w(f_{j+1})$, i.e. $w(T^{**}) = w(T^*)$.
Thus T^{**} is min-weight spanning tree and it contains a *longer* initial segment of the edges of T, than T^* did.

Repeating this procedure at most $(n-1)$-times, we transform any min-weight spanning tree into T.