Extremal Combinatorics — Wi 2015–16

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Exercise Sheet 10

Due date: 12:30, Jan 13th, at the beginning of lecture.
Late submissions will be covered by loving doodles of set pairs.

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Using the theorem of Alon, Babai and Suzuki that we saw in lecture (the generalised version of the non-uniform modular restricted intersection theorem), deduce the non-uniform non-modular Ray-Chaudhuri–Wilson theorem:

If L is a set of s integers, and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\} \subseteq 2^{[n]}$ is L-intersecting, then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{0}$.

Exercise 2 Let p be a prime, and $n \geq p^2 - 1$ an integer. Define the graph $R_{n,p}$ to have vertices $\binom{[n]}{p^2-1}$, and an edge between two vertices $F_1, F_2 \in \binom{[n]}{p^2-1}$ if and only if $|F_1 \cap F_2| \equiv -1 \mod p$.

(i) Show that $R_{n,p}$ has no clique or independent set of size $\binom{n}{p-1} + 1$.

(ii) By taking $n = p^3$, show that there is some constant $c > 0$ and an infinite sequence of values of t for which $R(t,t) \geq t^{\frac{c \log t}{\log \log t}}$.

Exercise 3 Extend Sperner's Theorem by showing that if $\mathcal{F} \subset 2^{[n]}$ does not contain any k-chains, then $|\mathcal{F}| \leq M_{n,k-1}$, where, for any $0 \leq s \leq n+1$,

$$M_{n,s} = \sum_{i=\lceil \frac{n-s}{2} \rceil}^{\lceil \frac{n+s}{2} \rceil - 1} \binom{n}{i}$$

is the sum of the s largest binomial coefficients.

Exercise 4 In this exercise we will give a different proof of Sperner's Theorem. Given $0 \leq k \leq n$, let $V_k = \binom{[n]}{k}$. For $0 \leq k \leq n-1$, let G_k be the bipartite graph on $V_k \cup V_{k+1}$, with an edge from $F \in V_k$ to $F' \in V_{k+1}$ if and only if $F \subset F'$.

(i) Show that when $k < \frac{n}{2}$, there is a matching $M_k : V_k \hookrightarrow V_{k+1}$ in G_k.

(ii) Deduce that when $k \geq \frac{n}{2}$, there is a matching $M_{k+1} : V_{k+1} \hookrightarrow V_k$ in G_k.

(iii) Partition $2^{[n]}$ into $\binom{n}{\lceil n/2 \rceil}$ chains and deduce the result of Sperner's Theorem.