Extremal Combinatorics Wi 2015–16
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Exercise Sheet 11

Due date: 12:30, Jan 20th, at the beginning of lecture.
Late submissions will be mailed to Donald Trump.[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Given some $n \in \mathbb{N}$ and $\delta_1, \delta_2 \in \mathbb{R}_{>0}$, let $S \subseteq \mathbb{R}^n$ be a set of points such that any two distinct points $s_1, s_2 \in S$ have Euclidean distance $d(s_1, s_2) = \|s_1 - s_2\|_2 \in \{\delta_1, \delta_2\}$.

(i) By constructing an appropriate linearly independent set of polynomials, show that $|S| \leq \frac{1}{2}(n+1)(n+4)$.

(ii) For $\delta_1 = \sqrt{2}$ and $\delta_2 = 2$, show that one can have $|S| \geq \binom{n}{2}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S11.html`.][2]

Exercise 2 We say a family of sets $\mathcal{F} \subseteq 2^{[n]}$ is *distinguishing* if for every $x, y \in [n]$ with $x \neq y$, there is some set $F \in \mathcal{F}$ such that $|F \cap \{x, y\}| = 1$. We say that $\mathcal{F}$ is *strongly distinguishing* if for every $x, y \in [n]$ there are sets $F_1, F_2 \in \mathcal{F}$ such that $x \in F_1 \setminus F_2$ and $y \in F_2 \setminus F_1$.

(i) Show that the smallest distinguishing family has size $\lceil \log_2 n \rceil$.

(ii) Show that the smallest strongly distinguishing family has size m, where m is the smallest integer such that $n \leq \binom{m}{\lfloor m/2 \rfloor}$.

[1] Only if sufficient postage is attached.

[2] It was brought to our attention that some students are unusually adept at reading text backwards, and were thus inadvertently reading some of our earlier hints. We have tried to resolve this situation by posting the hints as QR codes on another webpage. If you can accidently read QR codes, then you are probably a cyborg, and presumably have bigger issues to deal with than hints to these homework exercises.

Exercise 3 Given some graph F and some n, suppose we assign to each edge $e \in E(K_n)$ a vector $\vec{v}_e$ in such a way that for every copy F' of F in K_n (that is, $F' \cong F$ and $F' \subset K_n$), there is a relation

$$\sum_{e \in E(F')} \alpha_e \vec{v}_e = \vec{0}$$

with $\alpha_e \neq 0$ for all $e \in E(F')$. Let $W = \operatorname{span}\{\vec{v}_e : e \in E(K_n)\}$.

(i) Prove that, under these assumptions, $\operatorname{wsat}(n, F) \geq \dim W$.

(ii) Find an appropriate assignment of vectors to show $\operatorname{wsat}(n, K_3) = n - 1$.

Exercise 4 Show that there cannot be a skew version of the non-uniform[3] Bollobás set-pairs inequality. More specifically, for every $n \in \mathbb{N}$, find sequences of finite sets $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ such that

1. $A_i \cap B_i = \emptyset$, and
2. $A_i \cap B_j \neq \emptyset$ for every $i > j$,

but $\sum_{i=1}^{m} \binom{|A_i|+|B_i|}{|A_i|}^{-1} \geq n + 1$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S11.html`.]

[3] "Ununiform" would be a funny-looking word.