Extremal Combinatorics Wi 2015–16

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Exercise Sheet 12

Due date: 12:30, Jan 27th, at the beginning of lecture.
Late submissions will retracted to their boundaries.[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 In this exercise we will use the Kruskal–Katona theorem to strengthen the LYM inequality. First we will introduce some notation. Given numbers $k \geq 1$ and $m \geq 0$, let $KK(m,k)$ be the minimum size of the shadow of a family of m sets of size k guaranteed by the Kruskal–Katona theorem. That is, if $m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_s}{s}$ for some $a_k > a_{k-1} > \ldots > a_s \geq s$, then $KK(m,k) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_s}{s-1}$.

(i) In the *colexicographic* order on $\binom{\mathbb{N}}{k}$, we say $A < B$ if and only if $\max(A \Delta B) \in B$; informally, sets with larger elements come later. We write $\mathcal{C}(m,k)$ for the set family given by the first m sets in the colexicographic order on $\binom{\mathbb{N}}{k}$. Show that the bound in the Kruskal–Katona Theorem is tight by observing that $\partial\left(\mathcal{C}(m,k)\right) = \mathcal{C}(KK(m,k), k-1)$.

(ii) Strengthen the LYM inequality by proving the following statement about antichains. Given a vector $(a_0, a_1, \ldots, a_n) \in \mathbb{N}^{n+1}$, set $w_n = a_n$, and, for every $0 \leq k \leq n-1$, set $w_k = KK(w_{k+1}, k+1) + a_k$. There is then an antichain $\mathcal{A} \subseteq 2^{[n]}$ with exactly a_k sets of size k for every $0 \leq k \leq n$ if and only if $w_1 \leq n$ and $w_0 \leq 1$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S12.html`.]

Exercise 2 Given a set family $\mathcal{F} \subseteq \binom{[n]}{k}$, define its *$\ell$-shadow* to be

$$\partial_\ell(\mathcal{F}) = \left\{ E \in \binom{[n]}{\ell} : E \subset F \text{ for some set } F \in \mathcal{F} \right\}.$$

(i) For $0 \leq \ell < k$ and $m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_s}{s}$, where $a_k > a_{k-1} > \ldots > a_s \geq s$, determine $KK_\ell(m,k)$, the smallest possible size of the ℓ-shadow of a k-uniform set family $\mathcal{F}$ of size m.

(ii) Deduce the Erdős–Ko–Rado theorem: if $n \geq 2k$, the largest intersecting family in $\binom{[n]}{k}$ has size $\binom{n-1}{k-1}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S12.html`.]

[1] Discontinuously, of course.

Exercise 3 We define the *Kneser graph* $KG(n,k)$ to have vertices $V = \binom{[n]}{k}$, with edges $F_1 \sim F_2$ if and only if $F_1 \cap F_2 = \emptyset$. Observe[2] that $KG(5,2)$ is the well-known Petersen graph[3].

(i) For all $0 \le k \le n$, determine the chromatic number of the Kneser graph, $\chi(KG(n,k))$.

Given a graph G, let $\mathcal{I}(G)$ be the set of its independent sets. The *fractional chromatic number* $\chi_f(G)$ is defined as the minimum $r \in \mathbb{R}$ for which one may assign non-negative real numbers $x_I \ge 0$ to every independent set $I \in \mathcal{I}(G)$ such that $\sum_{I \in \mathcal{I}(G)} x_I = r$, subject to the constraint that for every vertex $v \in V(G)$, $\sum_{I \ni v} x_I \ge 1$.

(ii) Show that for any N-vertex graph G, $\frac{N}{\alpha(G)} \le \chi_f(G) \le \chi(G)$.

(iii) When $n \ge 2k$, show that $\chi_f(KG(n,k)) = \frac{n}{k}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S12.html`.]

Exercise 4 Consider the two statements below.

(BU) For any continuous map $f : S^d \to \mathbb{R}^d$, there is some $x \in S^d$ such that $f(x) = f(-x)$.

(SC) If $S^d = U_0 \cup U_1 \cup \ldots \cup U_d$, where for each $1 \le i \le d$, U_i is either open or closed, then there is some $0 \le j \le d$ such that U_j contains a pair of antipodal points $\{x, -x\}$.

In lecture we showed (BU) $\Rightarrow$ (SC). Show that they are in fact equivalent by proving (SC) $\Rightarrow$ (BU).

[2] This is just to check that you have the definition correct, and to sate your mathematical curiosity, and is not for credit.

[3] A respected combinator once told me that the Petersen graph is the only graph that "may not be ugly."