Extremal Combinatorics Wi 2015–16
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Exercise Sheet 13

Due date: 12:30, Feb 3rd, at the beginning of lecture.
Late submissions will be bisected by a hyperplane.

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 This exercise consists of the following two parts.

(a) Let $K \subset \mathbb{R}^2$ be a compact set. Show that there exist two hyperplanes h_1 and h_2 that together divide K into four parts of equal area.

(b) Recall that we say a hyperplane h bisects a finite point set A if $|h^+ \cap A|, |h^- \cap A| \leq \left\lfloor \frac{1}{2} |A| \right\rfloor$. One might think that it would instead make sense to count any point on the hyperplane with weight $\frac{1}{2}$ in each half-space. That is, you could say h bisects A if $|h^+ \cap A| + \frac{1}{2} |h \cap A| = \frac{1}{2} |A| = |h^- \cap A| + \frac{1}{2} |h \cap A|$. Show that, with this definition of bisection, there are finite sets of points A_1 and A_2 in $\mathbb{R}^2$ that cannot be simultaneously bisected by a line.

Exercise 2 Prove directly[1] that if A_1 and A_2 are disjoint sets of n points in $\mathbb{R}^2$ such that $A_1 \cup A_2$ is in general position, there is a perfect matching from A_1 to A_2 such that the straight line segments between the matched pairs are pairwise disjoint.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S13.html`.]

Exercise 3 To define a triangulation formally, one requires the notion of a *simplicial complex*. A triangulation of a topological space X is then a simplicial complex K with a homeomorphism $g : K \to X$.

A *face* of a simplex is the convex hull of some non-empty subset of its vertices. For instance, the faces of a tetrahedron (the 3-dimensional simplex) are the tetrahedron itself, its four triangles, its six edges and its four vertices. Note that a face of a simplex is itself a simplex (usually of lower dimension).

A *simplicial complex* is then a union of simplices (of various dimensions) such that if S is a simplex in the complex, all of its faces are also in the complex, and for any two non-disjoint simplices S_1 and S_2 in the complex, $S_1 \cap S_2$ is a face of both S_1 and S_2.

[1] Do not use the Ham Sandwich Theorem or any of its variants.

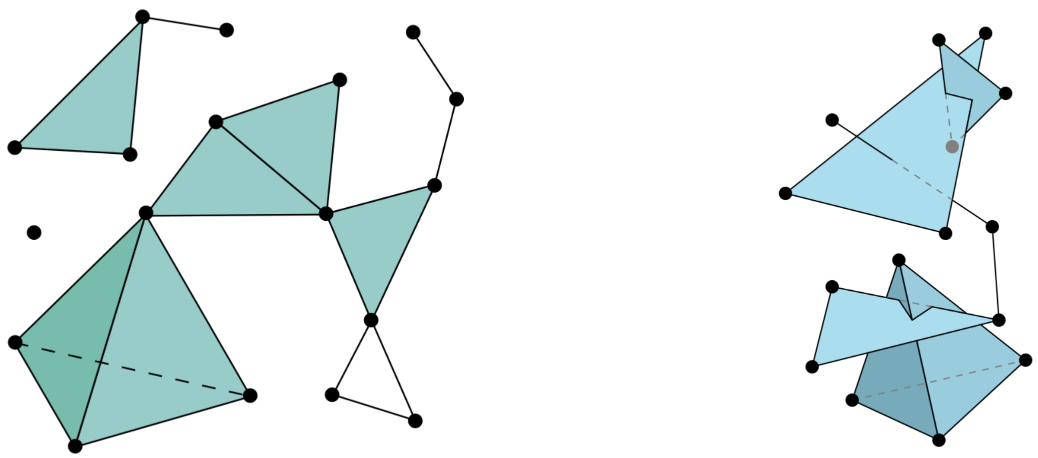

Figure 1: Left: a simplicial complex. Right: not a simplicial complex. [Wikipedia]

One can then define an *abstract simplicial complex* by mapping each simplex in the complex to the set of vertices of the simplex. However, the geometry now disappears, and we can state the definition in purely combinatorial terms. An abstract simplicial complex is a (non-uniform) hypergraph $\mathcal{F}$ with the property that if $F \in \mathcal{F}$ and $\emptyset \neq E \subseteq F$, then $E \in \mathcal{F}$.[2]

The f-vector of an abstract simplicial complex $\mathcal{F}$ is the (infinite[4]) vector $(f_0, f_1, f_2, \ldots)$, where f_i denotes the number of sets of size $i+1$ in $\mathcal{F}$.

We finally reach the exercise: given a vector $\vec{f} = (f_0, f_1, f_2, \ldots)$, determine necessary and sufficient conditions on the coordinates f_i for $\vec{f}$ to be the f-vector of some abstract simplicial complex.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S13.html`.]

Exercise 4 Show that between any n points in $\mathbb{R}^2$, there can be at most $\frac{\sqrt{2}}{2} n^{\frac{3}{2}} + n$ pairs of points with distance 1.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S13.html`.]

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[2] In other words, $\mathcal{F}$ is a down-set.[3]

[3] We hyphenate this word because its opposite should be 'up-set' and not 'upset'.

[4] But with only finitely many non-zero entries.