Extremal Combinatorics Wi 2015–16

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Exercise Sheet 14

Due date: 12:30, Feb 10th, at the beginning of lecture.
Late submissions will be mauled by a bear and left for dead.

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 In this exercise we will prove a slightly stronger version of the 2-dimensional case of Sperner's Lemma. Let S be a triangle, and suppose its vertices are coloured 1, 2 and 3 in clockwise order. Now consider any subdivision of S with a legal colouring of the new vertices.

For each subtriangle in the subdivision, we will assign a label to each of its three edges. Processing the colours of the vertices in clockwise order, give an edge label $+1$ if the colours of its endpoints are $(1, 2)$, $(2, 3)$ or $(3, 1)$. Give it label -1 if the colours are $(2, 1)$, $(3, 2)$ or $(1, 3)$. Give it label 0 if the colours are $(1, 1)$, $(2, 2)$ or $(3, 3)$.

Note that as an internal edge of the subdivision is contained in two subtriangles, it will receive two labels: one for each subtriangle it is in.

By considering the sums of all of the labels, show that the subdivision does not just contain a rainbow triangle, but it contains a rainbow triangle with vertices of colour 1, 2 and 3 in clockwise order.

Exercise 2 Let G be an $(2k)$-partite graph, with each part having n vertices, of maximum degree Δ.

(i) Show that if $n > 2\Delta - \frac{\Delta}{k}$, then G must have an independent transversal.

(ii) Show that this is best possible: construct a $(2k)$-partite graph with parts of size $2\Delta - \left\lceil \frac{\Delta}{k} \right\rceil$ and maximum degree Δ that has no independent transversals.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S14.html`.]

Exercise 3 We say a graph G has *degeneracy* d, written $\mathrm{degen}(G) = d$, if d is the smallest integer for which there is an ordering of the vertices such that every vertex has at most d edges to previous vertices. Let $\delta(G), \Delta(G)$ and $\chi(G)$ denote the minimum degree, maximum degree and chromatic number of G respectively.

(i) Show that for every graph G, $\delta(G) \leq \mathrm{degen}(G) \leq \Delta(G)$.

(ii) Prove that $\mathrm{degen}(G) = \Delta(G)$ if and only if G has a $\Delta(G)$-regular connected component.

(iii) Prove that $\chi(G) \leq \mathrm{degen}(G) + 1$.

Exercise 4 Let B^d denote the closed unit ball in $\mathbb{R}^d$; that is,

$$B^d = \left\{ \vec{x} \in \mathbb{R}^d : \sum_{i=1}^{d} x_i^2 \leq 1 \right\}.$$

Let $f : B^d \to B^d$ be a continuous function. Brouwer's Fixed Point Theorem states that f must have a *fixed point*; that is, there is some $\vec{x} \in B^d$ such that $f(\vec{x}) = \vec{x}$.

Since B^d can be continuously and reversibly deformed into the standard d-simplex Δ^d, where

$$\Delta^d = \left\{ \vec{x} \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i = 1, \text{ and } x_i \geq 0 \text{ for all } i \right\},$$

it is equivalent to show that any continuous map $f : \Delta^d \to \Delta^d$ has a fixed point. Use Sperner's Lemma to prove this version of Brouwer's Fixed Point Theorem.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2015-16/hints/S14.html`.]