Extremal Combinatorics Wi 2015–16

Tibor Szabó
Shagnik Das

Exercise Sheet 2

Due date: 12:30, Oct 28th, at the beginning of lecture.
Late submissions will be turned into papier-mâché.

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 In this exercise you should use an Erdős–Szekeres-type argument to improve our first bound on the hypergraph Ramsey numbers.

(i) Show that for $k \geq 2$ and $s, t \geq k+1$,

$$R^{(k)}(s,t) \leq R^{(k-1)}(R^{(k)}(s-1,t), R^{(k)}(s,t-1)) + 1.$$

(ii) Deduce the bound $R^{(3)}(4,4) \leq 19$.[1] You may cite any ordinary (2-uniform) Ramsey numbers without proof.

Exercise 2 By considering a random colouring, show that

$$R^{(k)}(t,t) \geq (1-o(1))(t/e)2^{\frac{1}{k}\binom{t-1}{k-1}} \geq (1-o(1))2^{c_k t^{k-1}},$$

where c_k is some constant that only depends on k.

Exercise 3 Prove the two following statements about points in general position in the plane (i.e. no 3 points lie on a common line).

(i) Any set of 5 points have a subset of 4 points in convex position.

(ii) Any set of n points *not* in convex position contain a subset of 4 points not in convex position.

[1] Amazingly, $R^{(3)}(4,4)$ is the *only* known[2] Ramsey number $R^{(k)}(s,t)$ when $k \geq 3$ and $s,t > k$. It turns out that $R^{(3)}(4,4) = 13$; a monochromatic-$K_4^{(3)}$-free construction on 12 vertices was given in by Isbell 1969, and in 1991 McKay and Radziszowski used computers to find the tight upper bound.

[2] As of January 2014, at any rate, but surely if there were any new results in the last 21 months, they'd have been all over the news, just like the Large Hadron Collider people[3] always are.

[3] That should be read as "(Large Hadron Collider) people", not "Large (Hadron Collider) people".

Exercise 4 The hypergraph Ramsey theorem shows (in its infinite extension) that, for any $k \geq 1$, if you red-/blue-colour all k-sets of natural numbers, $\binom{\mathbb{N}}{k}$, you can find an infinite set $S \subset \mathbb{N}$ such that $\binom{S}{k}$ is monochromatic.

What happens if we instead colour the *infinite* subsets of $\mathbb{N}$? Given an infinite set S, let $\binom{S}{\omega}$ denote the set of all infinite subsets of S. If we red-/blue-colour $\binom{\mathbb{N}}{\omega}$, must we find an infinite set $S \subset \mathbb{N}$ such that $\binom{S}{\omega}$ is monochromatic?

Bonus[4]: Note that things can go horribly wrong in the uncountable setting! Find a red-/blue-colouring of $\binom{\mathbb{R}}{2}$ for which there is no uncountably infinite $S \subset \mathbb{R}$ with $\binom{S}{2}$ monochromatic.

[4]Your reward for solving this bonus problem is our respect, which we hope you value far more than any other prize we could possibly offer.