Extremal Combinatorics Wi 2015–16
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Exercise Sheet 4

Due date: 12:30, Nov 11th, at the beginning of lecture.
Late submissions may be used as wallpaper.

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 In this exercise, you will complete the details of the Cherkashin–Kozik lower bound on the size of the smallest non-two-colourable k-graph. Let H be a k-graph with m edges. Recall that for each vertex $v \in V(H)$, we independently sample $x_v \sim U([0,1])$, a uniformly random number in $[0,1]$. We then order the vertices in increasing order of x_v, and run the greedy algorithm in that order. That is, we colour a vertex blue unless it is the last vertex in an all-blue edge, in which case it is coloured red.

(a) Consider the following events, where $\delta \in (0,1)$.

(i) $\mathcal{L}_e = \left\{ \forall v \in e : x_v < \frac{1}{2}(1-\delta) \right\}$ for some edge $e \in E(H)$.

(ii) $\mathcal{R}_f = \left\{ \forall v \in f : x_v > \frac{1}{2}(1+\delta) \right\}$ for some edge $f \in E(H)$.

(iii) $\mathcal{E}_{e,f} = \{ |e \cap f| = 1,$ the last vertex v of e is the first vertex of f, and $x_v \in \left[\frac{1}{2}(1-\delta), \frac{1}{2}(1+\delta) \right] \}$ for two edges $e, f \in E(H)$.

Show that $\mathbb{P}(\mathcal{L}_e) = \mathbb{P}(\mathcal{R}_f) = (1-\delta)^k 2^{-k}$ and $\mathbb{P}(\mathcal{E}_{e,f}) \leq \delta 2^{2-2k}$.

(b) Let $m = \beta 2^{k-1}$. Show that if $\beta(1-\delta)^k + \beta^2 \delta < 1$, then H is two-colourable.

(c) By choosing β and δ appropriately, show that there is some positive constant $c > 0$ such that $m_B(k) \geq c \left(\frac{k}{\ln k} \right)^{\frac{1}{2}} 2^k$.

Exercise 2 Suppose the natural numbers $\mathbb{N}$ are two-coloured. Van der Waerden's theorem implies the existence of arbitrarily long monochromatic arithmetic progressions. Must there exist an *infinite* monochromatic arithmetic progression $(a_0, a_0 + d, a_0 + 2d, a_0 + 3d, \ldots)$?

Exercise 3 Show the following strengthened version of Van der Waerden's theorem: for every r and k, there is some n such that whenever $[n]$ is r-coloured, there is a k-AP $(a_0, a_1, \ldots, a_{k-1})$ such that all the a_i's and the common difference d are monochromatic.

[Hint (to be read backwards): esU naV red s'nedreaW meroeht ot trats htiw a yrev gnol citamorhconom citemhtira noissergorp.]

Exercise 4 In the Van der Waerden k-AP game, two players alternately claim integers in $[n]$. When a player has claimed all the elements of some k-AP, he or she wins the game. If all elements in $[n]$ have been claimed without either player having completed a k-AP, the game ends in a draw. Show that when n is sufficiently large (in terms of k) the first player has a winning strategy.