Extremal Combinatorics Wi 2015–16
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Exercise Sheet 5

Due date: 12:30, Nov 18th, at the beginning of lecture.
Late submissions will [fill in the most terrible thing you can think of here[1]].

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Your mission, should you choose to accept it[2], is to show that the Turán graph $T_{n,t}$ has the most edges among all n-vertex t-colourable graphs.

(i) How many edges does the complete t-colourable graph $K_{n_1,n_2,\dots,n_t}$ have?

(ii) Show that if $K_{n_1,n_2,\dots,n_t}$ maximises the number of edges when there are n vertices in total, then $|n_i - n_j| \le 1$ for all $1 \le i,j \le t$. Deduce that $n_i \in \left\{ \left\lfloor \frac{n}{t} \right\rfloor, \left\lceil \frac{n}{t} \right\rceil \right\}$ for all $1 \le i \le t$.

(iii) Show that for every $\varepsilon > 0$ there is a $\delta = \delta(t,\varepsilon) > 0$ such that if n is sufficiently large, and if the numbers $(n_i)_{i=1}^t$ satisfy $\sum_{i=1}^t n_i = n$ and $e(K_{n_1,n_2,\dots,n_t}) \ge \left(1 - \frac{1}{t} - \delta\right)\frac{n^2}{2}$, then $\left| n_i - \frac{n}{t} \right| \le \varepsilon n$ for all $1 \le i \le t$.

Exercise 2 Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^d$ be a collection of vectors with (Euclidean) norm $\|\vec{v}_i\| \ge 1$ for all $1 \le i \le n$. Show that there are at least $\frac{n(n-2)}{4}$ pairs $\{i,j\}$ with $i \ne j$ such that $\|\vec{v}_i + \vec{v}_j\| \ge 1$.

Bonus: Let X_1 and X_2 be two independent, identically distributed random vectors in $\mathbb{R}^d$. Show that $\mathbb{P}\left(\|X_1 + X_2\| \ge 1\right) \ge \frac{1}{2}\mathbb{P}\left(\|X_1\| \ge 1\right)^2$.

Exercise 3 In this exercise, you will provide a different proof of a slightly weaker upper bound for Turán's theorem: you will show $\mathrm{ex}(n, K_{t+1}) \le \left(1 - \frac{1}{t}\right)\frac{n^2}{2}$. For this, we will introduce the *Lagrangian* $\lambda(G)$ of a graph G.

Let G be a graph on n vertices, $v_1, v_2, \dots, v_n$. Suppose we are given some probability distribution $\vec{p} = (p_1, p_2, \dots, p_n)$. Let u and w be two random vertices, $u, w \in V(G)$, that are chosen independently, both according to $\vec{p}$. That is, $\mathbb{P}(u = v_i) = p_i$ for all $1 \le i \le n$, and similarly for w.

[1]Please don't actually fill anything in.
[2]Disclaimer: If you choose not to accept it, you will not get points for this exercise.

For any such distribution $\vec{p}$, let

$$\lambda(G,\vec{p}) = \mathbb{P}\left(\{u,w\} \in E(G)\right) = \sum_{\substack{(i,j)\in[n]^2: \\ \{v_i,v_j\}\in E(G)}} p_i p_j.$$

That is, $\lambda(G,\vec{p})$ is the probability that our random vertices u and w, chosen according to $\vec{p}$, are adjacent. The *Lagrangian* of G, $\lambda(G)$, is the maximum value of $\lambda(G,\vec{p})$ over all possible distributions.

$$\lambda(G) = \max\left\{\lambda(G,\vec{p}) : p_i \geq 0, \sum_i p_i = 1\right\}.$$

(i) Show that if G is an n-vertex graph with m edges, $\lambda(G) \geq \frac{2m}{n^2}$.

(ii) Show that for any graph G, there is a probability distribution $\vec{p}$ on its vertices such that $\lambda(G,\vec{p}) = \lambda(G)$ and for all $1 \leq i < j \leq n$, if $p_i > 0$ and $p_j > 0$, then $\{v_i, v_j\} \in E(G)$. In other words, G has an optimal probability distribution that is supported on a clique.

(iii) Show that $\lambda(K_t) = 1 - \frac{1}{t}$.

(iv) Deduce that if G is an n-vertex K_{t+1}-free graph, then $e(G) \leq \left(1 - \frac{1}{t}\right)\frac{n^2}{2}$.

Exercise 4 For this exercise you may use the fact that, as $m \to \infty$, any $r \in [m]$ can have at most $2^{(1+o(1))\log m/\log\log m} = m^{o(1)}$ divisors $s|r$.

Given a set $A \subset \mathbb{Z}$, we define the *sumset* as $A + A = \{a + a' : a, a' \in A\}$.

(i) Show that if $\{1^2, 2^2, 3^2, \ldots, n^2\} \subseteq A + A$, then $|A| \geq \left\lfloor \sqrt{2n} \right\rfloor$.

(ii) Show that, as $m \to \infty$, for any number $r \in [m]$ there are at most $m^{o(1)}$ pairs $x, y \in \mathbb{N}$ such that $r = x^2 - y^2$.

(iii) Improve the bound from (i) by showing that for every $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$ and $\{1^2, 2^2, 3^2, \ldots, n^2\} \subseteq A + A$, then $|A| \geq n^{\frac{2}{3}-\varepsilon}$.