Extremal Combinatorics Wi 2015–16
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Exercise Sheet 6

Due date: 12:30, Nov 25th, at the beginning of lecture.
Late submissions will used to stuff a turkey.

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let $G = (V, E)$ be a graph, and let $A, B \subset V$ be two disjoint non-empty sets of vertices, of sizes a and b respectively. Let $X \subset A$ be a uniformly random set of k vertices, where $1 \le k \le a$ is fixed, and let $Y \subset B$ be an independent uniformly random set of ℓ vertices, for some fixed $1 \le \ell \le b$.

(i) Show that $\mathbb{E}[d(X, Y)] = d(A, B)$; that is, the average density over subsets is the density of the pair (A, B) itself.

(ii) Deduce that, in order to check whether or not the pair (A, B) is ε-regular, it suffices to check if $|d(X, Y) - d(A, B)| \le \varepsilon$ for every pair (X, Y) with $X \subset A$ of size precisely $\lceil \varepsilon a \rceil$ and $Y \subset B$ of size precisely $\lceil \varepsilon b \rceil$.

Exercise 2 Let (A, B) be an ε-regular pair of density d, and let $s \in \mathbb{N}$ be fixed. Let $Y \subset B$ be fixed. Show that, provided $(d - \varepsilon)^{s-1} |Y| \ge \varepsilon |B|$,

$$\left|\{(a_1, a_2, \ldots, a_s) \in A^s : |Y \cap (\cap_{i=1}^{s} N(a_i))| < (d - \varepsilon)^s |Y|\}\right| \le s\varepsilon |A|^s ,$$

where $N(a)$ is the neighbourhood of a vertex a. Hence only very few s-tuples of vertices in A have fewer than expected common neighbours in a large set Y.

[Hint (to be read backwards): noitcudnI si a s'rotanibmoc tseb dneirf!]

Exercise 3 The half-graph H_n is a bipartite graph with parts $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, and edges $E = \{\{a_i, b_j\} : i \le j\}$. Given $\varepsilon > 0$, and any even $k \in \mathbb{N}$, find an explicit ε-regular partition of H_n with k parts (not counting the exceptional set) when $n \ge k/\varepsilon$. How many irregular pairs does your partition have?

Bonus: Show that for every $\varepsilon > 0$ there is some constant $c = c(\varepsilon) > 0$ such that if $k \ge 2$ and n is large enough, any equipartition of H_n into k parts must have at least ck irregular pairs.

Exercise 4 In this exercise you will show that a random bipartite graph is ε-regular with high probability, thus validating our intuitive understanding that ε-regular pairs "look random". Fix some $\varepsilon > 0$ and some probability $p \in [0, 1]$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be the two vertex sets. For all $1 \leq i, j \leq n$, let the edge $\{a_i, b_j\}$ belong to our random graph G with probability p, independently of all other edges. Show that

$$\mathbb{P}\left((A, B) \text{ is } \varepsilon\text{-regular}\right) \to 1 \quad \text{as} \quad n \to \infty.$$

You may use the Chernoff bound, which in particular implies that if a random variable $X \sim \mathrm{Bin}(n, p)$ is binomially distributed with parameters[1] n and p, then for any $t \geq 0$,

$$\mathbb{P}\left(|X - np| \geq t\right) \leq 2 \exp\left(-2t^2/n\right).$$

[1] The n and p here are not related to the n and p from the statement of the problem.