Extremal Combinatorics Wi 2015–16
Tibor Szabó
Shagnik Das

Exercise Sheet 7

Due date: 12:30, Dec 2nd, at the beginning of lecture.
Late submissions will be punished to the fullest extent of the law.

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Show that for every $\varepsilon > 0$ there is some $\delta = \delta(\varepsilon) > 0$ such that if G is an n-vertex graph with fewer than δn^4 copies of K_4, then G can be made K_4-free by removing at most εn^2 edges.

Exercise 2 An *induced matching* M in a graph G is a matching such that for any two edges $\{u, v\}$ and $\{x, y\}$ of the matching M, $\{u, x\}, \{u, y\}, \{v, x\}$, and $\{v, y\}$ are not edges in the **host graph** G. Let $f(k, n)$ be the maximum number of edges a graph G can have if it is the union of k induced matchings; that is, $E(G) = \cup_{i=1}^{k} M_i$, where each M_i is an induced matching in G.

Show that for any $\varepsilon > 0$, if n is sufficiently large, $f(k, n) \leq 4\varepsilon n^2 + 2\varepsilon kn$ by applying the following steps:

(i) Apply the Regularity Lemma to G, and throw away at most $4\varepsilon n^2$ edges to find a subgraph $G' \subset G$ where every edge is contained in an ε-regular pair of density at least 2ε.

(ii) Let M' be an induced matching in G'. Let $I = \{i : |V(M') \cap V_i| \geq \varepsilon |V_i|\}$, where the V_i are the parts of the ε-regular partition of G. Show that $V(M') \cap (\cup_{i \in I} V_i)$ forms an independent set in G'.

(iii) Deduce that $|V(M')| \leq 4\varepsilon n$, and obtain the desired bound on $f(k, n)$.

Exercise 3 Let $r_3(n)$ be the size of the largest 3-AP-free subset $A \subset [n]$, and let $f(k, n)$ be the function from Exercise 2. Give an alternative proof of Roth's Theorem by constructing an appropriate bipartite graph to show $r_3(n) \leq f(n, 5n)/n$.

Exercise 4 Roth's Theorem states that if $A \subset [n]$ is 3-AP-free, then $|A| = o(n)$. The following exercise shows that this bound is not too far from the truth — there are 3-AP-free sets that are nearly linear in size.

(i) Given $d, m, r \in \mathbb{N}$, show that if $X = \{(x_1, x_2, \ldots, x_d) \in [m]^d : \sum_{i=1}^d x_i^2 = r\}$, then $A = \{\sum_{i=1}^d x_i(2m+1)^{i-1} : (x_1, x_2, \ldots, x_d) \in X\} \subset [(2m+1)^d]$ is 3-AP-free.

(ii) Show that one may choose m, d, and r appropriately to obtain a 3-AP-free subset A of $[n]$ of size $|A| = n \exp(-c\sqrt{\log n}) = n^{1-o(1)}$ (where $c > 0$ is some constant).