Extremal Combinatorics Wi 2015–16
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Exercise Sheet 8

Due date: 12:30, Dec 9th, at the beginning of lecture.
Late submissions will be forgiven[1], for 'tis the season for that kind of thing.

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if n is sufficiently large, and G is an n-vertex graph with $e(G) \geq \left(\frac{1}{4} + \varepsilon\right) n^2$, then G contains at least δn^3 triangles.

Exercise 2 A *corner* in $[n]^2$ is a triple of points of the form $\{(x, y), (x + h, y), (x, y + h)\}$ for some positive x and y and some non-zero h.[2] Using the Triangle Removal Lemma, show that for every $\varepsilon > 0$, if n is sufficiently large, then any $A \subset [n]^2$ with $|A| \geq \varepsilon n^2$ must contain a corner.

Exercise 3 Let G be a graph with vertices $V = \{v_1, v_2, \ldots, v_n\}$. Form a new graph $\mu(G)$ by taking a copy of G, and adding vertices $\{u_1, u_2, \ldots, u_n, w\}$ with the following edges:

- For every edge $\{v_i, v_j\} \in E(G)$, add the edges $\{u_i, v_j\}$ and $\{v_i, u_j\}$.
- For every $1 \leq i \leq n$, add the edge $\{u_i, w\}$.

For example, $\mu(K_2) = C_5$.

(i) Show that if G is triangle-free, then $\mu(G)$ is also triangle-free.

(ii) Show that $\chi(\mu(G)) = \chi(G) + 1$.

(iii) Deduce that for every k, there is a triangle-free graph G_k with $\chi(G_k) = k$.

Exercise 4 Suppose G is a graph with $\chi(G) = k$, and let $V = V_1 \cup V_2 \cup \ldots \cup V_k$ be a proper k-colouring of G (where each vertex in V_i receives colour i).

(i) Show that for every $1 \leq i \leq k$, there is a vertex $v_i \in V_i$ such that v_i has a neighbour in V_j for every $j \neq i$.

(ii) Deduce that G has at least $\binom{k}{2}$ edges.

(iii) If G is also triangle-free, improve this lower bound to $e(G) \geq \frac{3}{4}k^2 - k$.

[1] They will still not earn points, though.
[2] In other words, a corner is an axis-aligned isosceles right triangle.