Extremal Combinatorics — Wi 2015–16
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Exercise Sheet 9

Due date: 12:30, Dec 16th, at the beginning of lecture.
Late submissions will be used to wrap Christmas presents.[1]

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 This exercise concerns intersecting families $\mathcal{F} \subset 2^{[n]}$.

(i) Show that every maxim**al**[2] intersecting family in $2^{[n]}$ has size 2^{n-1}.

(ii) Show that when n is even, $2^{[n]}$ contains at least $2^{\frac{1}{2}\binom{n}{n/2}}$ different intersecting families with 2^{n-1} sets (that is, of maximum possible size).[3]

Exercise 2 Prove the lemma from the proof of the Erdős–Ko–Rado theorem. That is, if $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, show that any cyclic permutation π of $[n]$ can have at most k sets $F \in \mathcal{F}$ in cyclic order.

Exercise 3 Recall that the Erdős lower bound on the Ramsey number $R(t,t)$ did not provide an explicit construction. We saw that the Turán graph $T_{(t-1)^2,t-1}$ gives an explicit lower bound $R(t,t) > (t-1)^2$. Here we seek to provide a larger construction.

Construct a graph G whose vertices are the triples of elements in $[t-1]$. Put an edge between triples F and F' if and only if $|F \cap F'| = 1$. Show that G has no clique or independent set of size t, and deduce that $R(t,t) > \binom{t-1}{3}$.

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[1] Unless they are *very* late, in which case they may be used for birthday presents throughout the year.
[2] In lecture, we proved this for every maxim**um** intersecting family.
[3] Here we think of the set families as being labelled, so you do not need to worry about different families being isomorphic to each other. For example, a star with centre 1 is different from a star with centre 2.

Exercise 4 Neighbouring Oddtown is the little-known town of Evenoddertown. Curiously enough, this town also has a population of n people, who also enjoy making clubs. However, to distinguish themselves from their neighbours in Oddtown, the people of Evenoddertown make their clubs according to the following two rules:

1. Every club should have an even number of members.
2. Any two different clubs should have an odd number of common members.

What kind of restrictions do these modified rules provide?

(i) Show that there can be at most $n + 1$ clubs.

(ii) Improve the bound in (i) to show that if n is odd, there can be at most n clubs, while if n is even, there can be at most $n - 1$ clubs.

(iii) Construct sets of clubs to show the bounds in (ii) are best possible.

[Hint (to be read backwards): tahW si eht noisnemid fo eht ecaps eht citsiretcarahc srotcev naps? wohS taht yna noitaler neewteb eht srotcev tsum evlovni lla eht srotcev, dna osla na ddo rebmun fo srotcev.]