Exercise Sheet 9 - Extended Version[1]

Due date: 12:30, Dec 16th, at the beginning of lecture.
Late submissions will be used to wrap Christmas presents.[2]

You should try to solve all of the exercises[3] below, but clearly mark which two solutions you would like us to grade - each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Introduction   Stuart Dent[4] found himself torn between conflicting emotions. On the one hand, his family were soon about to head over to visit his grandmother, Susie Dent, for their annual Christmas dinner. He rather liked his grandmother[7], and her reputation as a chef was well-earned[8], so this trip was something he was greatly looking forward to. All was not well with the world, though, for on the last day of school before this Christmas holiday, Stuart discovered that he had, most unfortunately, failed his maths exam.[9] He

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[1]"Did we not," you sigh in exasperation, "say that the exercise sheets were too long? Should you not," you try to reason, "be truncating the exercise sheets, rather than extending them further?" You did, and yes, I should. However, unlikely though it may seem, there is a method to my madness. While Mathematics is a most beautiful subject in her own right, and you would no doubt be happy at this point simply to solve purely mathematical problems, I speak from personal experience when I say that further down the road, there may come a day when Mathematics for the sake of Mathematics may not be enough. In the midst of proving the Graceful Labelling Conjecture, you may put down your iPad Pro and ask yourself, "Why am I doing this? Will labelling these trees gracefully really make a difference?" It is precisely to avert these potential mid-life crises that we like to occasionally give you applied problems, submitted to us from real-world people looking for answers to their everyday combinatorial problems. It boosts the morale to know that one's work is not just of interest to the combinator next door, or the geometer one floor down, or even the theoretical computer scientist across the road, but also to the stranger you pass on the street, who may not personally care for Mathematics, but whose life is enhanced by her nevertheless. Thus, while it will remain our normal practice to distill these real-world problems into pure mathematical exercises for you to solve on a weekly basis, we hope you will appreciate our occasionally revealing the true story behind the homework problems you so happily receive.

[2]Unless they are *very* late, in which case they may be used for birthday presents throughout the year.

[3]Exercise 2 and 3 only have one part each, but we have still labelled them with a (i), to make it clearer what the actual exercise is.

[4]Brief biography: Stuart Arthur Dent, aged 14, only son of Arthur Dent and Trillian Astra. Currently enrolled in Year 9 of the Cottington Middle School, where he is not overly fond[5]of his fellow classmates. That is all there is to say, really, because 14-year-olds have rarely accomplished enough to merit a longer biography, and Stuart was, in this sense, not exceptional.

[5]His not-overly-warm feelings towards his classmates were not due to a generally displeasant disposition, but rather because of their tendency[6]to make fun of his name, often calling him SAD or Stu Dent. One may argue that Stu Dent is hardly offensive, and in fact the canonical abbreviation of his full name, but it is more in the *way* they said it, with the mean little chuckle to follow, and the fact that they continued calling him that despite his obvious displeasure.

[6]Anyone who has spent enough time around middle schoolers will readily attest to the fact that, no matter how poor their grasp of Measure Theory, Quantum Mechanics, or Neoclassical French Literature, they are remarkably quick to discover mean-spirited nicknames.

[7]A liking that was very mutual, for he was Susie's only grandchild. While Stuart reaped the benefits of this situation, though, Arthur often found himself being berated by Susie, who had quite been looking forward to having a granddaughter as well.

[8]Her alphabet soup was once described as a national treasure.

[9]It would appear that while 42 may be the answer to life, the universe, and everything, it was not the answer to every question on that test.

had not yet shared this bad news with his parents, and he did not quite know how to break it to them, for he did not want to get scolded before he had the internal support of his grandmother's food.

Thus Stuart held his tongue as he climbed into the back of the family's car[10], and forced a smile on his face as his parents starting singing the usual assortment of Christmas songs from the front seats. His mind drifted, as it was wont to do, while he stared out the window, and it was only when they were drawing into his grandmother's retirement community, an arrival announced by a friendly welcoming sign[11], that he snapped back into the present.[12] No sooner had his father parked than Stuart rushed out of the car, sprinting into the drab concrete building that housed his grandmother's apartment, for he was still young enough to enjoy being the first to ring the doorbell.

His parents had just caught up to him by the time his grandmother opened the door, wearing the Christmas-themed apron they had presented her the year before. She was enveloped in a bear hug from Stuart, followed by his father Arthur, although his mother Trillian limited herself to a forced smile and a lukewarm[13], "Hi Susie!" As they filed through the door and settled into the living room, Susie told them how excited she was to have them over, and how for the last few days she had done little other than count down the days to their visit, eliciting great laughter from Stuart and his parents.[14]

"Mom," asked Arthur, with a mixture of concern and perplexity etched on his large and expressive face, "is everything okay? Normally your home smells fantastic when we arrive, but it doesn't sound like you have anything on the stove right now." Hearing these words, the brave face Susie had been putting on for her family collapsed, and her voice started to quaver. "No, son," she replied, "things are not alright. I have not been able to start cooking yet." Before she could continue, a clearly distressed Stuart, whose visions of mince pies were quickly dissipating, cried, "Why not?!"

"Well," she explained, "I have not been able to decide whom to invite to this year's dinner. Without knowing the guest list, how can I possibly prepare an appropriate menu?" The confession out in the open, Susie's stiff upper lip collapsed, and she started to weep. Trillian's frostiness towards her mother-in-law was melted by this open display of emotion, and she lent a comforting handkerchief and suggested, "why don't you tell us what the problem is? I am sure we will be able to help, and get through this together."

Exercise 1 Cheered by this show of support from an unexpected corner, Susie gratefully accepted the handkerchief and dried her eyes. She explained, "due to our medical advances and the resulting aging population, Oddtown has been getting more and more

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[10] A trusty and reliable Ford Prefect that had served the family faithfully over the years, racking up considerable mileage with a minimum of fuss.

[11] "Welcome to Oddtown," it read, "Population: n, Clubs: n." A strange thing to advertise, thought Stuart, as they sped past the sign, and that was the end of his thoughts on the sign. Perhaps it was the speed at which the sign flew by, or perhaps it was the joyous anticipation of being reunited with his grandmother, or perhaps it was even the naïveté of youth that prevented him from further contemplation. Had they been driving slower, or had they not a sumptious meal to look forward to, or had Stuart been a little older and more familiar with the woes of the world, he might have reflected sadly on the state of the sign, which looked like it had last been painted in 1969. Its once bright and cheerful colours had faded into grey, a fact that some may have taken in with a sense of foreboding.

[12] Well, what was then the present, although it now may be the past - I have not been paying careful attention to my choice of tense here.

[13] The reason for this tension between mother-in-law and daughter-in-law was that Trillian always imagined that Susie blamed her for the lack of granddaughters, while Susie imagined that Trillian had only married her son to one day inherit her riches. Neither of these unfortunate assumptions were grounded in fact, but it would be a few years before they would realise this, leading to a dramatic improvement in their relationship.

[14] They were always amused when Susie would find a way to shoehorn in a reference to the quiz show that had made her a household name.

crowded. We now have 2^n people staying here, which is more than I can afford to cook for - I can cook for at most half that number. However, there are so many ..."

"Hold on, Grandma," Stuart interjected, "the sign outside said the population of Oddtown is only n, which is much smaller than 2^n." As he added that final remark, he smiled proudly at his father, hoping that this display of some mathematical knowledge would lessen the blow of the failed exam. "Oh," said Susie, dismissively, "that silly old sign hasn't been updated in ages. Anyhow, we have no time to waste, so let's try to stay on topic and avoid these tangential remarks that provide little information, and only serve to try and maintain some semblence of consistency throughout this story."[15]

"As I was saying," Susie continued, "I can only afford to invite at most 2^{n-1} people, although the fewer people I have to invite, the better. It's not been a great harvest this year."

"Well, that seems simple enough," Arthur said with relief. "We can make this a family affair, dinner for just the four of us."

"Unfortunately not," Susie countered. "Unless people have a good reason for not wanting to come themselves, they will get offended if I do not invite them. You know how I am - it would upset me greatly to offend anyone."

"That's trouble," said Trillian, "because why on Earth would anyone not want to come to your dinner? Your cooking is the talk of town," she added[16]. "Surely then you will have to find a way to feed all 2^n people."

"That's very kind of you to say, my dear," responded Susie, "but thankfully there is a way out of this predicament. As you may recall from the sign, Oddtown has n clubs.[17] Unfortunately, interclub rivalries are worse than they have ever been before, and so people are only friendly towards those with whom they share a club in common."

"So what you are saying, Mom," said Arthur, thoughtfully, trying to explain the situation to himself[18] out loud, "is that if I take a person, and look at the set of clubs she is a member of, and I have another person with a disjoint set of clubs, then they will not like each other, and you then cannot invite both of them to your dinner."

"That is exactly right, my dear son. You always were so clever," Susie beamed, ruffling Arthur's hair affectionately. "I am a member of all n clubs, so everyone would be happy to come to my dinner, provided there is no one on the guest list with a disjoint set of clubs." She was about to continue, when she saw Stuart had his hand raised in the air. "Is there something you would like to say, Stuart?"

"I just realised that you have 2^n people here in Oddtown, and there are exactly 2^n possible different sets of clubs a person can be a member of. What a funny coincidence," remarked Stuart, relieved at finally getting to speak after so long.

"Why, yes, I suppose that is true," was Susie's reply, "I wonder how I never realised that before. In fact, now that I think about it, no two people have the same set of clubs. So for every possible set of clubs, there is exactly one person who is a member of precisely those clubs."

"So what we need to do," said Trillian, with the germ of a plan developing in her mind, "is find as small a set of people to invite, with the property that every pair of them have some club in common, but any person not invited has a disjoint set of clubs with at least one person on the guest list. I suppose you could say that we are looking for the smallest

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[15]"Fine," muttered Stuart, a little stung at his grandmother's harsh tone. "I'll just keep my `tangents' to myself," he grumbled in a footnote-y way. His mumbling was ignored by the grown-ups.

[16]With a tinge of jealousy, recalling all the times Arthur had suggested that he preferred the food he ate growing up to what they now eat at home.

[17]Surely they could now afford to have 2^n clubs, thinks Stuart, but he wisely keeps his thoughts to himself, for fear of being told off again.

[18]And, fortunately for us, anyone who might be reading this conversation.

maximal[19] intersecting family of club membership sets."

"Precisely, love," said Susie, "but how small can a maximal intersecting family be?"

(i) Deliver some bad news to Grandma Susie by showing her that every maximal intersecting family in $2^{[n]}$ has size 2^{n-1}.

Susie seemed to take the bad news surprisingly well - though she was disappointed that she would have to cook for so many people, she was at least relieved to have a mathematical proof that she could not do otherwise. However, having settled the question of *how many* people to invite, there remained the tricky decision of *whom* to invite. "Well," she said cheerfully, "at least there cannot be very many intersecting families of size 2^{n-1}, right? That should make the choice easier!"

(ii) Deliver further bad news to Grandma Susie by pointing out that if n is even[20], then $2^{[n]}$ contains at least $2^{\frac{1}{2}\binom{n}{n/2}}$ different intersecting families of size 2^{n-1}.

Once again, Susie Dent met this latest setback with an admirably positive attitude. "I suppose in some sense that is a relief," she explained, "because with so many options to choose from, I am bound to upset some people. However, the inevitability of this disappointment absolves me of personal blame. Let us just pick some possible guest list arbitrarily then."

So resolved, they chose one of the many maximal intersecting families to serve as their guest lists, and hastily wrote up the invitations. Our little protagonist Stuart was tasked with running door-to-door to deliver the invitations by hand.[21]

Exercise 2 While Stuart was busy with his errand, the grown-ups moved to the kitchen, aiming to get the cooking underway. As Susie pulled her notebook filled with her special Christmas recipes out of a drawer, she called out, "Trillian, dear, would you be so kind as to pull out my spice rack? It's in the cupboard next to the sink."

Trillian was so kind as to do as requested, and before long the spice rack was placed on the kitchen counter. It was a beautiful hand-crafted wooden spice rack, circular in shape, designed so that the rack could be spun around the centre for easy access. On the circumference of the rack were n slots, each filled with a bottle with a different spice.[22]

However, much to Arthur's surprise, Susie quickly took all of the spice bottles out of the rack. "Mother," he asked, "why are you taking all of the spices out?"

"Well," she explained, "I need to arrange the spices for maximum efficiency. Each of my recipes, of course, calls for exactly k different spices to be used.[23] However, if the spices are all out of order, then I have to keep spinning the rack around, thus increasing the chances of my making a mistake and using the wrong spice. That simply won't do."

"I see," mused Arthur, continuing his convenient habit of explaining things out loud, as if to an unseen audience, "you want to place the spices in such a cyclic permutation that the set of spices used in each recipe appear next to each other, in a contiguous interval

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[19]"By which I mean," she explained in an aside to Stuart, who seemed a little confused, "that the family is intersecting, but you cannot add any other set to get a larger intersecting family."

[20]Which, by Murphy's Law, it is.

[21]What with Stuart and his family having visited Oddtown for Christmas dinners over the last several years, the residents had these annual snapshots of Stuart growing up, and many were quite enamoured with him. The hope was that they would be so happy to see him that they might overlook the lateness of the invitation.

[22]If you are having difficulty picturing this spice rack, imagine something along the lines of http://www.amazon.de/Emsa-Galerie-Spice-Carousel-Spices/dp/B000A8BKVU/, but handcrafted out of wood.

[23]Any fewer and the food would be bland, and any more and you would lose the intrinsic flavour of the ingredients themselves. Neither of these are mistakes a masterchef like Susie Dent would make.

on the rack. But wouldn't that limit the number of recipes you could make? Are there not only n intervals of length k in any cyclic permutation?"

"That's right," came the reply, "but n recipes should be more than enough. However, there is an additional wrinkle. These are my Christmas-themed recipes, so they share some similarities. In fact, it appears that every pair of my recipes have at least one spice in common.[24] Of course, no two different recipes use exactly the same set of k spices either.[25] At least we have $n \geq 2k$, so there is a lot of space to order the spices appropriately."

She turned to Trillian and said, "I want to make these $k+1$ recipes," pointing to certain pages in her notebook. "Can you arrange the spices so that the $k + 1$ intersecting sets of k spices are all in cyclic order on the spice rack?"

"I am afraid not," replied Trillian, instantly. "Under these restrictions, you could only make k recipes."

> (i) Show that when $n \geq 2k$, if $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, then any cyclic permutation π of $[n]$ can have at most k sets $F \in \mathcal{F}$ in cyclic order.

"Oh dear," sighed Susie, "I suppose I will have to make do with one fewer recipe then. This is quite the conundrum!" They all laughed.[26] After much deliberation, she decided that they can do without the Brussels sprouts, and settled on a menu of k items.[27] With all the decision-making complete, it was time to get to work, and the cooking began in earnest. By this time, young Stuart had finished making the rounds, having invited everyone lucky enough to be selected for the guest list, and returned in time to help out with the kitchen chores as well.

Some hours later, the cooking was complete, the (rather large) dining table was set, and the guests were streaming in. After some light conversation over drinks, everyone took their place around the table, and the dishes were served, one by one. Revelling in the pleasant company while stuffing themselves with food, nobody looked out of the window. Had they done so, they would have seen some curious lights appear in the sky above the neighbouring village of Evenoddertown. However, in the apartment of Susie Dent, these lights went unnoticed, and everyone continued to make merry.

Exercise 3 In Evenoddertown, however, the mysterious lights did not go unnoticed. In fact, a sense of panic quickly spread through the inhabitants of Evenoddertown, as the lights were revealed to belong to an Unidentified Flying Object piloted by three aliens.[28] Using some form of intergalactic translation software, they addressed the terrified inhabitants below: "Greetings, Earthlings. If you wish to live, present us with a graph on $\Omega(t^3)$ vertices that contains neither a clique nor an independent set on t vertices. You have 24 hours to give us this graph. If you fail, you will be exterminated. Thank you, and have a good day." Their message delivered, the spaceship vanished[29], leaving the bewildered citizens of Evenoddertown below.

Fearing that the appearance (and sudden disappearance) of these aliens could provoke disorder and riots, the Mayor of Evenoddertown called a town meeting in their town hall.

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[24] That is, she reasons, sensible, for having two dishes with disjoint sets of spices would provide for a jarring and unpleasant sensation. They couldn't both be Christmassy.

[25] That, she reasons, would provide for a dull and boring dining experience.

[26] For once again, Susie cleverly slipped in a reference to her famous quiz show. She has such a way with words!

[27] A Christmas menu with a curiously large number of seafood dishes, but Arthur and Trillian wisely decided not to question Susie's decision.

[28] If you are having difficulty picturing this, see here: http://tinyurl.com/jpkp2wa.

[29] Either by means of some advanced cloaking technology, or into another dimension - it happened too quickly to tell.

"Order, order," he cried, quieting the rowdy crowd. "Does anyone have any idea what those aliens were going on about? Cliques? Independent sets? Graphs?"

Fortunately for him, and for the other residents of Evenoddertown, one of Evenoddertown's many clubs was a Combinatorics Club. The president of Combinatorics Club, Erdős Turán, stepped up to the podium, and gave his fellow citizens a quick introduction to Ramsey Theory.

"So what you are saying," said Felix Christophson[30], "is that this other Erdős guy constructed these Ramsey graphs on $2^{t/2}$ vertices? Can we not just give the aliens one of those?"

"The problem is," explained Erdős[31], "that Erdős's[32] construction is random. If we use his construction, there is a possibility that we would fail, and then get exterminated. Is that a risk you are willing to take?"

"No," shouted everyone in unison.

"Then we must find an explicit construction of a Ramsey graph. I know that the Turán graph $T_{(t-1)^2,t-1}$ has no clique or independent set of size t, but that only has $O(t^2)$ vertices. We need something bigger."[33]

"Say, I have an idea." Everyone turned around in their seats to see who had spoken up, and were surpised to find the voice belonged to the young combinator Nagy. "What if we take a graph whose vertices are all sets of three elements in $[t-1]$, and put an edge between two triples if and only if they have an intersection of size 1?"

"That seems like a very peculiar construction," commented the bemused Erdős Turán. "Why would that work?"

(i) Explain why Nagy's construction has no clique or independent set of size t, and deduce the constructive lower bound $R(t,t) > \binom{t-1}{3}$.

Satisfied with Nagy's construction, the citizens of Evenoddertown relaxed, and went about their normal business, no longer concerned with imminent extermination. Sure enough, nobody panicked when the aliens returned the following day. The Mayor met them as they arrived, and presented them with Nagy's construction. The aliens spent some time testing the graph, and, unable to find a large clique or independent set, announced that they were satisfied and that they would not be exterminating anyone. Upon hearing this, the citizens gave a sarcastic cheer, and then confronted the aliens. They angrily shouted, "what gives you the right to come here and threaten us with extinction?"

"Extermination," corrected one of the aliens[34], "not extinction. You see, we are stranded and a long way away from our home galaxy. Our spaceship uses Ramsey graphs as fuel, but we had run out and needed some new ones. Our research on your planet suggested that you would be more likely to act out of self-preservation than for humanitarian reasons, and so we made up the threat of extermination. In actual fact, without any Ramsey graphs to fuel our phasers, we would not have been able to harm you at all. Nevertheless, we apologise for the deception, and thank you for your assistance. Farewell!"

Exercise 4 The aliens were about to set their engines to hyperdrive, when they were faced with flashing sirens. "Oi there," yelled a policeman[35], "don't you move. I hereby place you under arrest!"

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[30]Who was not a member of the Combinatorics Club, and will not feature again in this story, and hence will not have his backstory fleshed out any further.

[31]Evenoddertown's Erdős Turán, not the famous Paul Erdős he was named after.

[32]The famous Paul Erdős, not Evenoddertown's Erdős Turán.

[33]'I've heard that before,' thought Erdős Turán, but he kept this thought to himself.

[34]It has a name, and a gender, but we have neither the characters nor the pronouns to express either accurately here.

[35]Superintendent Sher Lockholmes, of the Somerset County Police.

"Whatever for?" asked the alien, whose research on our planet did not include the crucial information that it should refuse to speak without a lawyer present. "We already told you that we were not actually going to exterminate anyone."

"Exterminate?" The policeman appeared confused, for he had only just driven into Evenoddertown, and knew nothing of this empty threat. "This town has only two rules: that every club should have an even number of members, and any two clubs should have an odd number of members in common. You are violating both rules, for not only are there three of you in your Spaceship Club, but you have zero members in common with all the other clubs."

"Those are some very peculiar rules," the alien observed. "Wouldn't that severely limit the number of clubs your town could have?" After thinking about it a little, it further said, "As far as I can see, if your town has n people, you wouldn't be able to have more than $n + 1$ clubs."

(i) Show that there can be at most $n + 1$ clubs under these rules.

Before anyone else could respond, the alien continued, "actually, it is even worse than that. If n is odd, then you could have at most n clubs, while if n is even, there could be at most $n - 1$ clubs. That is not very many clubs at all."

(ii) Improve the bound in (i) to these new bounds suggested by the alien.

It was now Erdős Turán's turn to look confused. "I do not see how you get those improved bounds," he professed. "Could you give us a hint?"

"Most certainly," the alien replied, helpfully. "tahW si eht noisnemid fo eht ecaps eht citsiretcarahc srotcev naps? wohS taht yna noitaler neewteb eht srotcev tsum evlovni lla eht srotcev, dna osla na ddo rebmun fo srotcev."[36]

Having decoded the hint and solved the problem, Erdős Turán was satisfied with the new bounds, and nodded contentedly. At this point, the Mayor spoke up, "can you really make so many clubs? We could only make two clubs under these rules - the Combinatorics Club and the Stamp Collectors' Club. To be honest, with so few offerings, life here in Evenoddertown has been pretty boring."

(iii) Show that the previous bounds are best possible, by constructing sets of clubs satisfying the two rules and matching the bounds from (ii), for both even and odd n.

Conclusion Having helped the Mayor organise many more clubs, thus drastically improving the quality of life of those in Evenoddertown, the aliens were viewed much more favourably by the citizens of Evenoddertown. They argued with the policeman, explaining that without having filed a club charter with the town hall, the three aliens and their "Spaceship Club" were not officially recognised, and hence were not subject to the rules of Evenoddertown. The policeman conceded that they had a fair point, and the aliens were allowed to return to their home galaxy.[37]

Meanwhile, back in Oddtown, Susie Dent and her guests remained blissfully unaware of the extraterrestrial visitors in the neighbouring town. The dinner party was a roaring success, the guests left happily, and Stuart and his parents helped Susie finish all the washing up. It was soon time for them to return home, and so they took their time saying their goodbyes. As was customary, Stuart went last, giving his grandmother another tight hug while saying, "So long, and thanks for all the fish."[38]

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[36]It would appear that its intergalactic translation software suffered a temporary glitch.

[37]Even without this legal defence, the punishment for violating the rules of Evenoddertown is exile, and so the aliens would have had to leave in any case.

[38]He said this out of politeness, but he really would have preferred some turkey instead.

Editorial remark Some of you might not have approved of the choice of font in this exercise sheet.[39] However, this exercise sheet is part of a long-term project to remove the stigma that I feel is unfairly attached to Comic Sans. I acknowledge that, like any other font, it is not appropriate for use in *every* context, but one should be able to use it occasionally without facing the violent backlash that most Comic Sans users are subject to in today's intolerant world. Thank you for your understanding.

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[39]Indeed, some of you may look to press charges: http://www.comicsanscriminal.com/.