Van der Waerden's Theorem

An r-coloring of a set S is a function $c : S \to [r]$.
A set $X \subseteq S$ is called monochromatic if c is constant on X.

Let $I\!N$ be two-colored.
Is there a monochromatic 3-AP?

Roth's Theorem says: YES, in the larger of the two color classes.

A weaker statement, not specifying in which color the 3-AP occurs:

Proposition In every two-coloring of $\left[2 \cdot (5 \cdot (2^5 + 1))\right]$ there is a monochromatic 3-AP.

What if we want a longer arithmetic progression?
Can we color the integers with two colors such that there is no monochromatic 4-AP?
Szemerédi's Theorem says NO.

How far must we color the integers to find an AP of length 4? Or k?

In order to prove something about this, we introduce more colors.

$W(r,k)$ is the smallest integer w such that any r-coloring of $[w]$ contains a monochromatic k-AP.

Theorem (van der Waerden, 1927) For every $k, r \geq 1$, $W(r,k) < \infty$.

Remark Consequence of Szemerédi's Theorem.

Proof of Van der Waerden's Theorem

Induction on k, the following statement:
"For all $r \geq 1$, $W(r,k) < \infty$"

$W(r,1) = 1$
$W(r,2) = r+1$
$W(r,3) = ?$

Suppose $W(r,k) < \infty$ for every $r \geq 1$.
Let us find an upper bound on $W(r,k+1)$ in terms of these numbers.

$$W(1,k+1) = k+1$$

$$W(2,k+1) \leq 2 \cdot (2W(2,k)) \cdot W(2^{2W(2,k)}, k)$$

$$W(3,k+1) \leq 2 \cdot 2 \cdot 2W(3,k) \cdot W(3^{2W(3,k)}, k) \cdot W(3^{2 \cdot (2W(3,k)) \cdot W(3^{2W(3,k)}, k)}, k)$$

For general r, define the (kind of fast growing) function $L_r : I\!N \to I\!N$,

$$L_r(x) = xW(r^x, k).$$

Then

$$W(r, k+1) \leq \underbrace{2L_r(\cdots 2L_r(2L_r(2L_r(1))))}_{r\text{-times}}.$$

We prove by induction on i, that no matter how the first $x_i = \underbrace{2L_r(\cdots 2L_r(2L_r(2L_r(1))))}_{i\text{-times}}$ integers are colored with r colors, there exists i monochromatic k-APs $a^{(j)}, a^{(j)} + d_j, \ldots, a^{(j)} + (k-1)d_j$, $1 \leq j \leq i$, each in different colors, such that $a^{(j)} + kd_j$ is the very same integer a for each j, $1 \leq j \leq i$.

Divide $[L_r(x_i)]$ into blocks of x_i integers. There are r^{x_i} ways to r-color a block. By the definition of $W(r^{x_i}, k)$, there is a k-AP of blocks with the same coloring pattern.

Let c_j be the color of the monochromatic k-AP $a^{(j)}, a^{(j)} + d_j, \ldots, a^{(j)} + (k-1)d_j$, for $1 \leq j \leq i$.

Case 1. If the color of $a = a^{(j)} + kd_j$ is one of these colors then there is a $(k+1)$-AP in this color and we are done.

Case 2. Otherwise the copies of a in the k blocks forms a monochromatic k-AP of color $c_{i+1} \neq c_j$, $1 \leq j \leq i$. We can form monochromatic k-APs in the other colors c_j: Take the copy of $a^{(j)} + (l-1)d_j$ from the l^{th} block.

These $i+1$ k-APs are monochromatic of $i+1$ distinct colors and would be continued in the same $(k+1)^{st}$ element. This element is certainly less than $2L_r(x_i)$.

After the rth iteration the colors run out, Case 2 cannot occur, and we have a monochromatic $(k+1)$-AP.
$\square$