Algorithmic Combinatorics — Wi 2016–17

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Solution to Exercise Sheet 15, Exercise 1

Exercise 1 Recall the statement of the Lovász Local Lemma we had in class.

> **Theorem 1** (Lovász Local Lemma). *Let $E_1, E_2, \ldots, E_m$ be events in some probability space. Let $d \in \mathbb{N}$ and $p \in [0,1]$ be such that, for every $i \in [m]$, we have*
>
> *(1) $\mathbb{P}(E_i) \le p$, and*
>
> *(2) there is a set $\Gamma(i) \subseteq [m] \setminus \{i\}$ of at most d indices, such that the event E_i is mutually independent of $\{E_j : j \in [m] \setminus (\Gamma(i) \cup \{i\})\}$.*
>
> *If $ep(d+1) \le 1$, then with positive probability none of the events E_i occur.*

In this exercise you will prove Theorem 1.

(a) Show that for any $i \in [m]$ and $J \subseteq [m] \setminus \{i\}$, we have $\mathbb{P}\left(E_i | \cap_{j \in J} E_j^c\right) \le ep$. You may use the estimate $(1 - 1/(d+1))^d \ge e^{-1}$.

(b) Deduce that $\mathbb{P}\left(\cap_{i \in [m]} E_i^c\right) \ge (1 - ep)^m > 0$.

Solution: For convenience, given a subset $S \subseteq [m]$ of the events, we let $E_S^c = \cap_{i \in S} E_i^c$ be the event that none of the events indexed by S occur.

(a) We wish to show that for any $i \in [m]$ and $J \subseteq [m] \setminus \{i\}$, we have $\mathbb{P}(E_i | E_J^c) \le ep$. We shall do so by induction on $|J|$. Let $I = \Gamma(i) \cap J$, so that E_i is mutually independent of $\{E_j : j \in J \setminus I\}$.

First, consider the case $I = \emptyset$ (which contains the base case, $|J| = 0$). In this case, E_i is independently of $\{E_j : j \in J\}$, and so $\mathbb{P}(E_i | E_J^c) = \mathbb{P}(E_i) \le p < ep$, as required.

Otherwise, we have

$$\mathbb{P}(E_i | E_J^c) = \frac{\mathbb{P}(E_i \cap E_I^c | E_{J \setminus I}^c)}{\mathbb{P}(E_I^c | E_{J \setminus I}^c)}.$$

For the numerator, $\mathbb{P}(E_i \cap E_I^c | E_{J \setminus I}^c) \le \mathbb{P}(E_i | E_{J \setminus I}^c) = \mathbb{P}(E_i) \le p$. For the denominator, suppose we have $I = \{i_1, i_2, \ldots, i_s\}$, where $s = |I| \ge 1$. We then have

$$\mathbb{P}(E_I^c | E_{J \setminus I}^c) = \prod_{\ell=1}^{s} \mathbb{P}(E_{i_\ell}^c | E_{i_1}^c, \ldots, E_{i_{\ell-1}}^c, E_{J \setminus I}^c) = \prod_{\ell=1}^{s} \left(1 - \mathbb{P}(E_{i_\ell} | E_{i_1}^c, \ldots, E_{i_{\ell-1}}^c, E_{J \setminus I}^c)\right).$$

For each factor in this product, we may apply the induction hypothesis, since $|J \setminus I| + \ell - 1 \leq |J \setminus I| + s - 1 = |J| - 1$. This shows each factor is at least $1 - ep$, and so the denominator is at least $(1 - ep)^{|I|}$.

Now observe that $|I| \leq |\Gamma(i)| \leq d$, and recall that $ep \leq \frac{1}{d+1}$, and so the denominator is at least $\left(1 - \frac{1}{d+1}\right)^d \geq e^{-1}$. This shows

$$\mathbb{P}(E_i|E_J^c) = \frac{\mathbb{P}(E_i \cap E_I^c|E_{J\setminus I}^c)}{\mathbb{P}(E_I^c|E_{J\setminus I}^c)} \leq \frac{p}{e^{-1}} = ep,$$

as required. This completes the induction.

(b) If we let $[0] = \emptyset$, we have

$$\mathbb{P}(E_{[m]}^c) = \prod_{i=1}^{m} \mathbb{P}(E_i^c|E_{[i-1]}^c) = \prod_{i=1}^{m} \left(1 - \mathbb{P}(E_i|E_{[i-1]}^c)\right) \geq (1 - ep)^m > 0,$$

where we use part (a) for the penultimate inequality.