Algorithmic Combinatorics — Wi 2016–17
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Solution to Exercise Sheet 15, Exercise 2

Exercise 2 Recall that the Ramsey number $R(k,k)$ is the smallest n such that any two-colouring of the edges of K_n must contain a monochromatic copy of K_k.

(a) By colouring edges randomly, show that if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. Deduce that $R(k,k) \geq \frac{1}{e\sqrt{2}}(1+o(1))k2^{k/2}$. [This is from Discrete Math I.]

(b) Obtain a $\sqrt{2}$-factor improvement of the result in (a) by 'correcting' a random colouring by removing a vertex from every monochromatic clique: show that for any integer n, $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$. Deduce that $R(k,k) \geq \frac{1}{e}(1+o(1))\, k2^{k/2}$.

(c) Improve the bound by yet another $\sqrt{2}$-factor with the Local Lemma: show that if $e\binom{k}{2}\binom{n-2}{k-2} 2^{1-\binom{k}{2}} \leq 1$, then $R(k,k) > n$. Deduce the bound $R(k,k) \geq \frac{\sqrt{2}}{e}(1+o(1))k2^{k/2}$.

Solution:

(a) Given n, and consider colouring the edges of K_n independently, uniformly at random. For a given set of k vertices, the probability they induce a monochromatic clique is $2^{1-\binom{k}{2}}$, since there are two possible colours, and each of the $\binom{k}{2}$ edges will be given that colour with probability $1/2$. As there are $\binom{n}{k}$ sets of k vertices, the expected number of monochromatic cliques of size k is $\binom{n}{k} 2^{1-\binom{k}{2}}$. By assumption, this is strictly less than 1, which is only possible if there is some edge-colouring of K_n without any monochromatic k-clique. Hence we must have $R(k,k) > n$.

To get a good lower bound on $R(k,k)$, we need to choose n as large as possible while $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ holds. We can bound the left-hand side by

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \left(\frac{ne}{k}\right)^k 2^{1-\binom{k}{2}} = 2\left(\frac{ne\sqrt{2}}{k2^{k/2}}\right)^k.$$

Thus if $n = \frac{2^{-1/k}}{e\sqrt{2}} k2^{k/2} = \frac{1}{e\sqrt{2}}(1+o(1))k2^{k/2}$, the above expression is equal to 1, and so we deduce $R(k,k) > \frac{1}{e\sqrt{2}}(1+o(1))k2^{k/2}$.

(b) For larger n, the expected number of monochromatic cliques will be large, so we cannot hope to find a monochromatic-k-clique-free K_n by taking a random edge colouring. However, if the number of monochromatic cliques is not *too* large, we can remove a vertex from each such clique to be left with a good colouring on a smaller (but not too small) number of vertices.

Indeed, if we start with n vertices, and remove one vertex from every monochromatic clique, we would on average be left with at least $n - \binom{n}{k}2^{1-\binom{k}{2}}$ vertices, showing there exists a monochromatic-k-clique-free graph of at least that size, and hence $R(k,k) \geq n - \binom{n}{k}2^{1-\binom{k}{2}} \geq n - 2\left(\frac{ne\sqrt{2}}{k2^{k/2}}\right)^k$.

As before, to get a good bound on $R(k,k)$, we should choose n to maximise this lower bound. Differentiating with respect to n and setting the derivate equal to 0, we find

$$2^{(3-k)/2}e\left(\frac{ne\sqrt{2}}{k2^{k/2}}\right)^{k-1} = 1,$$

or

$$n = \frac{2^{(k-3)/(2k-2)}e^{-1/(k-1)}}{e\sqrt{2}}k2^{k/2} \sim \frac{1}{e}(1+o(1))k2^{k/2},$$

which, after some calculation, gives the required bound on $R(k,k)$.

(c) For the final[1] improvement, we make use of the Lovász Local Lemma. Once again, we shall colour the edges of K_n independently and uniformly at random. For each set $K \subset V(K_n)$ of k vertices, let E_K be the event that k-clique on K is monochromatic. As before, we have $\mathbb{P}(E_K) = 2^{1-\binom{k}{2}}$.

The event E_K is determined by the edges supported on K, and hence is mutually independent of $\{E_{K'} : |K \cap K'| \leq 1\}$, since these events do not depend on any of the edges of K. Thus the number of events that E_K is not mutually independent of (including E_K itself) is at most $\binom{k}{2}\binom{n-2}{k-2}$, since we must choose some edge of K that they have in common, and then can choose the remaining $k-2$ vertices freely.

Taking these values as p and $d+1$ respectively, it follows from the Local Lemma that if

$$e\binom{k}{2}\binom{n-2}{k-2}2^{1-\binom{k}{2}} \leq 1,$$

then with positive probability our random colouring of K_n will not have any monochromatic k-cliques, and thus $R(k,k) > n$.

Asymptotically, the left-hand side is

$$ek^2\left(\frac{ne}{k-2}\right)^{k-2}2^{-k(k-1)/2} = \frac{ek^2}{2}\left(\frac{ne}{(k-2)2^{(k+1)/2}}\right)^{k-2}.$$

This will be smaller than 1 when $n = \frac{\sqrt{2}}{e}(1+o(1))k2^{k/2}$, as claimed.

[1] Indeed, despite turning 40 this year, this bound (due to Spencer) remains the best-known lower bound. While that may seem a long time without progress, I'd like to point out that its been a good 48 years since the lunar landing, in which time we still haven't got past first base with the moon.