Solution to Exercise Sheet 15, Exercise 4

Exercise 4 The city of London is surrounded by the M25 motorway, a circular road that directs traffic around the city without congesting its inner roads. It is approximately 110 miles long and, as per UK traffic regulations, has 30 streetlights per mile, and thus a total of 3300 lampposts.

To comply with recent environmental guidelines, the Mayor of London wants to illuminate the M25 with environmentally-friendly lightbulbs that will consume less power while maintaining adequate light coverage. To find the best lightbulb for the job, he commissions London's 300 different lighting firms to submit prototypes for evaluation.

Each firm provides a sample of 11 lightbulbs. To ensure that no firm has all its lightbulbs in a favourable stretch of the highway, all 3300 lightbulbs are mixed together and then placed, in some arbitrary order, in the M25's lampposts. The Mayor intends to keep these lightbulbs in place for a month and evaluate their efficiency before making a final decision about which lightbulb to use in the long-term.

Unfortunately, after a few days, he realises that this experiment is rather expensive, and decides the test has to be scaled down[1]. Thus one of each company's 11 lightbulbs will be switched off. However, in the interests of public safety, no two neighbouring lightbulbs should both be switched off, for fear of creating too long a dark stretch on the motorway.

Show that, regardless of how the lightbulbs were initially distributed, it is always possible to safely turn off one lightbulb from each company.

Solution: As we have not spent much time studying lightbulbs in this course, let us first rephrase the problem in more familiar terms. We shall make an auxiliary graph G, where the vertices of G are the lampposts, and edges in G correspond to neighbouring lampposts. Since the M25 is a circular road, it follows that G will be a cycle with 3300 vertices.

Now we have an arbitrary partition of the vertices, $V(G) = \sqcup_{i=1}^{300} V_i$, where V_i is the set of lightbulbs from the ith lighting firm. In particular, $|V_i| = 11$ for all i. We wish to turn off one lightbulb from each company; that is, some $x_i \in V_i$ for $1 \leq i \leq 300$. However, to do so safely, we must not turn off two neighbouring lightbulbs, and so $X = \{x_i : 1 \leq i \leq 300\}$ must be an independent set. In the parlance of graph theory, such a set X is often referred to as an *independent transversal*.

To show that an independent transversal exists, we apply the Lovász Local Lemma. For each i, $1 \leq i \leq 300$, let x_i be chosen independently and uniformly at random from V_i. As we want $X = \{x_i : 1 \leq i \leq 300\}$ to be an independent set, we will define the event

[1] An alternative would have been to raise taxes to fund the project, but he is a proud patriot, and, after a rather poor showing at the UEFA Euro 2016, decides his country can ill afford to surrender either of her two advantages over France: lower taxes and finer food.

E_e for each edge $e \in E(G)$ of both of the endpoints of e being selected in X. Since X consists of exactly one vertex from each class V_i, if the endpoints of e belong to the same class, the event E_e never occurs. Otherwise the endpoints belong to different classes and are chosen independently, each with probability $1/11$, and so the event E_e holds with probability $1/11^2 = 1/121$. Thus, setting $p = 1/121$, we always have $\mathbb{P}(E_e) \leq p$.

We must take a little care in determining the dependencies of the events E_e. Note that if the endpoints of e belong to classes V_i and V_j, then the event is determined solely by knowing x_i and x_j. Hence E_e is mutually independent of all events corresponding to edges spanned by $V \setminus (V_i \cup V_j)$.

As G is a cycle, every vertex is incident to 2 edges. Since $|V_i \cup V_j| \leq |V_i| + |V_j| = 22$ (the inequality appears because we might have $i = j$), there are at most $2 \cdot 22 = 44$ edges involving vertices in $V_i \cup V_j$. This includes the edge e itself, so we have $d + 1 \leq 44$.

By the Local Lemma, since $ep(d + 1) \leq 44e/121 < 1$, it follows that with positive probability, none of the events E_e occur.[2] In this case, X is an independent set in G, thus showing the existence of an independent transversal.

[2]It turns out that $40e/10^2 > 1$, and so if each company had only submitted 10 lightbulbs, we could not have used this proof to show that it would be possible to safely remove one of each of the companies' ten lightbulbs. However, it is a challenging (but fun!) exercise to show that even if every company had only submitted *four* lightbulbs, which were then distributed arbitrarily along a (shorter) circular road, you could still remove one of each companies' lightbulbs without removing two consecutive bulbs. A more general statement (with a beautiful proof) will be proven in next year's Extremal Combinatorics course.