Solution to Exercise Sheet 15, Exercise 5

Exercise 5 In the notes on the algorithmic Local Lemma, it is shown that for a k-uniform hypergraph $\mathcal{F}$ with n vertices and $\Delta(L(\mathcal{F})) \leq 2^{k-4}$, the expected number of recolourings in the algorithmic Local Lemma is $O(m \log m)$. With more careful analysis, show that this bound can be greatly improved to $O\left(\frac{n}{k} \log m\right)$.

Solution: Recall how the $O(m \log m)$ bound is obtained. We constructed a rooted ordered tree to keep track of the recolourings of the randomised colouring algorithm. By showing that, apart from the top-level calls, each random recolouring of the k elements of a set could be efficiently encoded in $k-1$ bits, we deduced that it is very unlikely that there should be many lower-level recolourings, and hence in expectation each top-level call leads to $O(\log m)$ lower-level recolourings.

We also showed that at the end of a top-level call for a set F, any set intersecting a set that was recoloured during the top-level call (including any set intersecting F) is properly coloured, and hence will not subsequently appear as a top-level call. In particular, each set can invoke a top-level call at most once, and hence there are at most m top-level calls, giving the $O(m \log m)$ bound.

However, the previous fact shows that the sets involved in the top-level calls must actually form a matching. As a matching of k-sets in $[n]$ can have size at most $\frac{n}{k}$, we get the improved $O\left(\frac{n}{k} \log m\right)$ bound on the number of recolourings.