Freie Universität Berlin
Institut für Mathematik
Algorithmic Combinatorics

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Exercise	1.	2.	3.	4.	5.	6.	Σ
Points							

MIDTERM EXAM

Show all your work and state precisely any theorems you use from the lectures.

This midterm exam is an excuse to revise what we have covered in the course so far, while also familiarising yourself with the format of the final exam. Completing this exam is entirely optional and will not earn you any credit, but if you would like to have it graded, please submit it by 2pm on the 3rd of January. We will discuss the solutions during the exercise class later that same day.

Exercise 1 [10 points]

Prim's Algorithm grows a spanning tree from a given vertex of a connected weighted graph G, iteratively adding the cheapest edge (ties broken arbitrarily) from a vertex already reached to a vertex not yet reached, finishing when all the vertices of G have been reached. Prove that Prim's Algorithm produces a minimum-weight spanning tree of G.

Exercise 2 [10 points]

(a) Show that every graph $G = (V, E)$ contains a set of vertices $S \subseteq V$ such that $2\alpha'(G) = |V| + |S| - o(G - S)$.

(b) Explain why the decision problem "Is there a matching of size k in G?" is in **NP** $\cap$ **co-NP**. (where G is an n-vertex graph and $k = k(n)$ is an integer that might depend on n.)

Exercise 3 [10 points]

(a) State and prove the Weak Duality Theorem for maximum weight matchings in $K_{n,n}$, defining the necessary terms.

(b) The following matrix gives the edge weights of a complete bipartite graph G on vertices $X = \{a, b, c, d, e\}$ and $Y = \{f, g, h, i, j\}$.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} f & g & h & i & j \end{array} \\ \begin{pmatrix} 4 & 4 & 4 & 3 & 6 \\ 1 & 1 & 4 & 3 & 4 \\ 1 & 4 & 5 & 3 & 5 \\ 5 & 6 & 4 & 7 & 9 \\ 5 & 3 & 6 & 8 & 3 \end{pmatrix} \end{array}$$

Use the Hungarian Algorithm to find a matching of maximum weight in G.

Exercise 4 [10 points]

(a) Define the list-colouring number $\chi_\ell(G)$ of a graph G.

(b) Prove that if G is planar, $\chi_\ell(G) \leq 5$.

Exercise 5 [10 points]

(a) Define the edge-chromatic number of a graph, and state Vizing's Theorem.

(b) Show that every graph G with maximum degree Δ has a $(\Delta + 1)$-edge-colouring with each colour class having either $\left\lfloor \frac{e(G)}{\Delta+1} \right\rfloor$ or $\left\lceil \frac{e(G)}{\Delta+1} \right\rceil$ edges.

Exercise 6 [10 points]

(a) Given a graph G and vertices $x, y \in V(G)$, define the parameters $\kappa'(x, y)$ and $\lambda'(x, y)$.

(b) Prove that $\kappa'(x, y) = \lambda'(x, y)$, carefully stating any theorems used.