

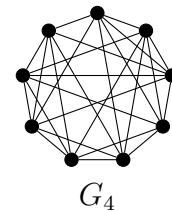
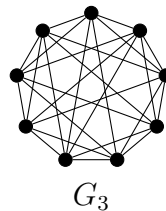
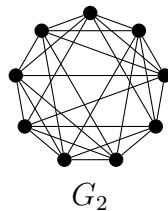
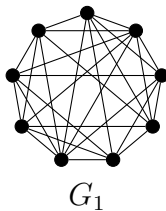
Exercise Sheet 0

Due date: Not for submission.

Solutions will be discussed on Oct 18th, in the exercise class.

The following exercises are designed to refresh your knowledge of some topics from Discrete Mathematics I and to wake you from the mathematical hibernation that is the summer holiday. You are encouraged to try to solve all of these problems, and should feel free to work on them with a partner or in a small group.

Exercise 1 Determine which pairs of the graphs below are isomorphic.



[Hint at <http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S00.html>.]

Exercise 2 Consider a complete weighted graph with vertices $\{1, 2, \dots, 7\}$, with the weight $w_{i,j}$ of the edge from i to j given in the matrix below.

$$W = (w_{i,j})_{1 \leq i, j \leq 7} = \begin{pmatrix} - & 5 & 3 & 1 & 5 & 2 & 0 \\ 5 & - & 4 & 7 & 1 & 4 & 9 \\ 3 & 4 & - & 3 & 6 & 2 & 3 \\ 1 & 7 & 3 & - & 2 & 2 & 2 \\ 5 & 1 & 6 & 2 & - & 5 & 6 \\ 2 & 4 & 2 & 2 & 5 & - & 6 \\ 0 & 9 & 3 & 2 & 6 & 6 & - \end{pmatrix}.$$

Find a spanning tree of minimum weight, where the weight of a tree is the sum of the weights of its edges.

Exercise 3 Suppose G is a graph with n vertices and m edges, where $m < \binom{n}{2}$.

- (a) Show that G can be disconnected by removing at most $\frac{2m}{n}$ vertices.
- (b) What is the maximum number of edges an n -vertex graph that is *not* 2-connected can have?

Exercise 4 An island kingdom¹ used coins for their financial transactions, and had a total of n different coins, one of value $\frac{1}{k}$ for each $1 \leq k \leq n$.² Asymptotically, what is the maximum total value a pile of coins can have if no subset of the coins is worth exactly 1?³

[Hint at <http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S00.html>.]

¹Distant in both space and time and forgotten to all but us readers of this exercise.

²This may seem rather peculiar set of denominations to you, but this is solely due to your decimal upbringing. The subjects of this kingdom were a more rational people, and they, no doubt, would laugh at our use of the coins $\frac{2}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{50}$ and $\frac{1}{100}$.

³This may seem a strange question to ask, and indeed the problem may never had arisen had Alice and her band of pirates not set foot on the Island of Mythos.

“Who is Alice?” you ask.⁴

Well, Alice, with an expensively exotic parrot permanently perched on her shoulder, led a fierce (but otherwise parrotless⁵) pirate crew consisting of Bob, Charles, Diana and Erik. Being pirates, they roamed the seas, and were very skilled at extracting money from merchants. However, after one raid, they were bitterly disappointed to find that the unlucky merchant had only a \$1 coin from Mythos to offer them.⁶

However, all was not lost, because Mythos had a very strong currency, so this \$1 was no mean amount. Still, the constitution of Alice’s band of pirates demanded that they divide the loot between them, which was an impossible task given that they had only one coin. Hence they set sail for the island, in search of a citizen who would provide them with some smaller denominations.

Having docked their ship, they soon came across a local lady, and asked her if she would do them the favour of exchanging their \$1 coin for some smaller change. She looked into her coin purse, but then told them that, unfortunately, she did not have enough coins to help them. Disappointed by her inability to help them, Alice, Bob, Charles, Diana and Erik conferred about whether or not to relieve this lady of her coin purse.

Erik said, “You know, I can’t really be bothered to rob her if we’re not even going to get \$1 out of it. I *just* cleaned my sword, and I’m not getting blood on it for next to nothing.”

“Hold on,” replied Diana, “we don’t know that she has less than \$1. For example, if she had one $\frac{1}{2}$ coin and two $\frac{1}{3}$ coins, she would have $\frac{7}{6}$, but still not be able to give us change for our \$1 coin. Maybe she is in fact very rich!”

And so our pirates set about trying to determine how much money they could gain by robbing this lady, giving rise to the exercise at hand (and also giving the lady in question ample time to make her escape).

⁴Figuring that Mythos must be the aforementioned island kingdom, that is the only question you ask.

⁵If you are puzzled by the discrepancy in parrot ownership, you may wish to consult Exercise 5 on Sheet 2 of last term’s Discrete Mathematics I course.

⁶As it turned out, the merchant was not as good at his trade as the pirates were at theirs, and he paid for his shortcomings with his life.⁷

⁷You might have thought that all of our pirates’ victims would have been killed, but Alice’s crew were staunch believers in sustainable piracy: by setting their merchants free with half of their money, they could earn more money to be stolen at a later date.

Exercise 5 In a country of population $2n + 1$, there are two candidates running for the office of president, Candidate B and Candidate G. It turns out this election is as close as can be — there are n citizens who support Candidate B, and $n + 1$ who support Candidate G.⁸

As per the constitution of the country, every citizen must cast a vote. The votes are then counted one-by-one, with a live total broadcast on national television. In how many ways can the votes be ordered so that Candidate B is never behind Candidate G until the very last vote is counted?

[Please turn over]

⁸Given the timing of this exercise, you might be excused for thinking that I was referring to the current US presidential election, and hoping that the future resident of the White House would not be decided by so slim a margin.

However, this exercise goes back a little further, to the 2000 Bush v Gore election. You may be a bit young to remember that contest well, but those were more civilised times, when the presidential debates were filled not with sleazy scandals but matters of policy, and when, ultimately, combinatorics⁹ was the deciding factor. In the end, Gore actually won the popular vote by a cool half million people, but, unfortunately for him, that half million did not live in Florida, whose 25 electoral college votes handed Bush the presidency.

Still, Gore realised he did not have to be POTUS to make the world a better place, and continued to use his public profile to raise awareness for climate change. These efforts culminated in his starring in the 2006 documentary “An Inconvenient Truth” and earned him the Nobel Peace Prize in the following year.

⁹Well, counting and recounting (not double-counting), but really, what’s the difference?

Exercise 6 A music producer wishes to produce a new album, with the idea of having several leading artists contributing songs.¹⁰ He thus prepares a list of n possible song titles, and mails it to the n most popular artists of the day, asking each artist to select at least three songs that they would be willing to work on.

- (a) Prove that if no song is selected four times or more, then the producer can put all n songs on his album, with each sung by a different artist.
- (b) Show by example that this may not be possible if some songs are selected four times.

¹⁰“What album is this?” you wonder, always looking to add to your collection, but also hoping for a short and succinct answer.

Recall, I begin, clearing settling into another absurdly long story, the documentary that was mentioned two footnotes ago. It was relatively well-received in its day, even winning an Oscar, and so the producers decided to host a gala this year to mark its ten-year anniversary. Al Gore received an invitation to this celebration and, reminded of this documentary he had featured in, turned to his social media manager, Christoph.

“I don’t get it,” he said, “we made this documentary a decade ago, won lots of awards, and yet the climate is worse than ever. Greenhouse gas emissions are at an all-time high, the ocean levels are endangering several island chains, and summers are hotter and deadlier than ever before. Why are people not taking action?”

Christoph, having heard of the gala a few days before, had anticipated this issue might be raised, and so whipped out his iPad to show his employer the data he had gathered. “I’m afraid, Mr. Gore, that your documentary did not reach a very wide audience. In theatres it was competing with blockbusters such as ‘The Departed’, ‘Casino Royale’ and ‘The Da Vinci Code’, not to mention ‘Borat’. Besides, our research shows that documentaries are not very popular among the current generation, so you should probably find another medium for your message.”

“I presume,” replied a disappointed Gore, “that you have some ideas you’d like to share with me.”

“Indeed I do,” said Christoph, for he was very good at his job. “For the greatest impact, you should make a guest appearance on ‘Game of Thrones’. However, the timeline is not favourable for us, as Westeros is bracing itself for the coming winter, which would make any discussion of global warming rather incongruous. Instead, I would suggest you produce a music album, because kids today like to listen to music. If you get the n leading stars of the industry to collaborate with you, you will reach as wide an audience as possible. As any advertiser knows, subliminal messaging is a really powerful tool, so any environmental references we insert should have a real impact.”

“That sounds like a capital idea,” responded Gore (who had by now put the 2000 electoral defeat behind him, and was free to use words such as ‘capital’ without painful flashbacks), “let’s get to work on it without delay.”

So they devoted themselves to the task of songwriting, producing such hits as ‘It’s getting hot in here (at the alarming rate of five degrees per century)’, ‘Your love keeps lifting me higher (which keeps me from drowning as the ice caps melt)’, ‘There’s a hole in my heart (and a much bigger one in the ozone layer)’ and ‘Lady, you’re one of a kind (just like the last-surviving male northern white rhino)’.

They also found many leading vocal celebrities were willing to contribute to this project (unfortunately Ice Cube was not available to sing ‘Help, I don’t want to become a puddle’), leading to its unfettered success and giving rise to greater global subconscious concern about the state of our planet.

And that, my friends, is the story behind “Al Gore Rhythms”.