Algorithmic Combinatorics   Wi 2016–17
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Exercise Sheet 0

Due date: Not for submission.
Solutions will be discussed on Oct 18th, in the exercise class.

The following exercises are designed to refresh your knowledge of some topics from Discrete
Mathematics I and to wake you from the mathematical hibernation that is the summer
holiday. You are encouraged to try to solve all of these problems, and should feel free to
work on them with a partner or in a small group.

Exercise 1   Determine which pairs of the graphs below are isomorphic.

G_1 G_2 G_3 G_4

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S00.html`.]
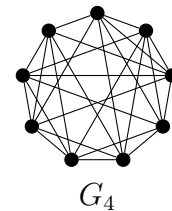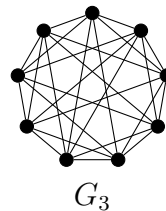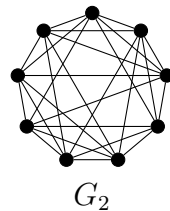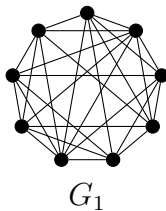
Exercise 2   Consider a complete weighted graph with vertices $\{1, 2, \ldots, 7\}$, with the weight
$w_{i,j}$ of the edge from i to j given in the matrix below.

$$
W = (w_{i,j})_{1 \le i,j \le 7} =
\begin{pmatrix}
- & 5 & 3 & 1 & 5 & 2 & 0 \\
5 & - & 4 & 7 & 1 & 4 & 9 \\
3 & 4 & - & 3 & 6 & 2 & 3 \\
1 & 7 & 3 & - & 2 & 2 & 2 \\
5 & 1 & 6 & 2 & - & 5 & 6 \\
2 & 4 & 2 & 2 & 5 & - & 6 \\
0 & 9 & 3 & 2 & 6 & 6 & -
\end{pmatrix}.
$$

Find a spanning tree of minimum weight, where the weight of a tree is the sum of the
weights of its edges.

Exercise 3   Suppose G is a graph with n vertices and m edges, where $m < \binom{n}{2}$.

(a) Show that G can be disconnected by removing at most $\frac{2m}{n}$ vertices.

(b) What is the maximum number of edges an n-vertex graph that is *not* 2-connected can
   have?

Exercise 4   An island kingdom used coins for their financial transactions, and had a total of n different coins, one of value $\$\frac{1}{k}$ for each $1 \leq k \leq n$. Asymptotically, what is the maximum total value a pile of coins can have if no subset of the coins is worth exactly 1?

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S00.html`.]

Exercise 5   In a country of population $2n + 1$, there are two candidates running for the office of president, Candidate B and Candidate G. It turns out this election is as close as can be — there are n citizens who support Candidate B, and $n + 1$ who support Candidate G.

As per the constitution of the country, every citizen must cast a vote. The votes are then counted one-by-one, with a live total broadcast on national television. In how many ways can the votes be ordered so that Candidate B is never behind Candidate G until the very last vote is counted?

Exercise 6   A music producer wishes to produce a new album, with the idea of having several leading artists contributing songs. He thus prepares a list of n possible song titles, and mails it to the n most popular artists of the day, asking each artist to select at least three songs that they would be willing to work on.

(a) Prove that if no song is selected four times or more, then the producer can put all n songs on his album, with each sung by a different artist.

(b) Show by example that this may not be possible if some songs are selected four times.