Algorithmic Combinatorics — Wi 2016–17

Shagnik Das — Tibor Szabó

Exercise Sheet 10

Due date: 14:00, Jan 17th, by the end of the lecture.
Late submissions will feed my fireplace.[1]

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Use the simplex algorithm to solve the following linear program.

$$\begin{array}{lrcrl} \text{maximise} & x_1 & + & 2x_2 & \\ \text{subject to} & -2x_1 & + & 2x_2 & \geq 2, \\ & 2x_1 & + & x_2 & \leq 2, \\ & & & x_1 & \in \mathbb{R}, x_2 \geq 0. \end{array}$$

Exercise 2 Dantzig originally suggested that the entering variable be the one with the largest coefficient r_i. By considering the following linear program, show that the simplex algorithm can indeed cycle with this pivot rule.

$$\begin{array}{lrcrcrcrl} \text{maximise} & 46x_1 & + & 43x_2 & - & 271x_3 & - & 8x_4 & \\ \text{subject to} & 2x_1 & + & x_2 & - & 7x_3 & - & x_4 & \leq 0, \\ & -39x_1 & - & 7x_2 & + & 39x_3 & + & 2x_4 & \leq 0, \\ & & & & & & & \vec{x} & \geq \vec{0}. \end{array}$$

Exercise 3 While running the simplex algorithm we arrive at a feasible basis B with simplex tableau $\mathcal{T}(B)$

$$\begin{aligned} \vec{x}_B &= \vec{p} + Q\vec{x}_N \\ z &= z_0 + \vec{r}^{\,T}\vec{x}_N \end{aligned}$$

has parameters $\vec{p} = (p_i)_{i\in B} \in \mathbb{R}^m$, $Q = (q_{i,j})_{i\in B, j\in N} \in \mathbb{R}^{m\times(n-m)}$, $\vec{r} = (r_j)_{j\in N} \in \mathbb{R}^{n-m}$ and $z_0 \in \mathbb{R}$. Suppose that this solution is not optimal, and we pivot to a basis $B' = B \setminus \{u\} \cup \{v\}$. Let $\mathcal{T}(B')$ be the new tableau, with parameters $\vec{p}', Q', \vec{r}'$ and z_0'. Find formulae for $p_i', q_{i,j}', r_i'$ and z_0' in terms of $p_i, q_{i,j}, r_i$ and z_0.

[1] Just kidding, I don't have a fireplace, but I'll burn them anyway.

Exercise 4 Suppose that while running the simplex algorithm, we perform a valid pivot from some basic feasible solution $\vec{x}_0$. Prove that the resulting solution $\vec{x}_1$ is also a basic feasible solution.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S10.html`.]

Exercise 5 Consider the linear program

$$\begin{array}{ll} \text{maximise} & \vec{c}^{\,T}\vec{x} \\ \text{subect to} & A\vec{x} = \vec{b}, \vec{x} \geq \vec{0}. \end{array}$$

(a) A set $X \subseteq \mathbb{R}^n$ is said to be *convex* if whenever $\vec{x}, \vec{y} \in X$ and $\alpha \in [0,1]$, we have $\alpha\vec{x} + (1-\alpha)\vec{y} \in X$ as well. Show that the set of feasible solutions to our linear program is convex.

(b) As the intersection of finitely many half-spaces, the feasible set is in fact a convex *polyhedron* in $\mathbb{R}^n$. A *vertex* of a polyhedron P is a point $\vec{v} \in P$ such that there is some vector $\vec{c} \in \mathbb{R}^n$ with $\vec{c}^{\,T}\vec{v} > \vec{c}^{\,T}\vec{y}$ for all $\vec{y} \in P \setminus \{\vec{v}\}$. Prove that $\vec{x}$ is a vertex of the feasible set for our linear program if and only if it is a basic feasible solution.

Exercise 6 Consider the linear program[2]

$$\begin{array}{lrl} \text{maximise} & \sum_{i=1}^{m} x_i & \\ & & \\ \text{subject to} & 2\sum_{i=1}^{k-1} x_i + x_k + x_{m+k} & = 2^k - 1, \text{ for all } 1 \leq k \leq m, \\ & \vec{x} & \geq \vec{0}. \end{array}$$

(a) Show that the maximum value is $2^m - 1$.

(b) Prove that for any feasible solution $\vec{x}$, we must have $x_k + x_{m+k} \geq 1$ for every $1 \leq k \leq m$.

(c) Deduce that, for any basic feasible solution $\vec{x}$ and for any $1 \leq k \leq m$, either x_k or x_{m+k} is a basic variable, but not both.

(d) Prove that this linear program has exactly 2^m distinct basic feasible solutions.

Bonus (5 pts) Show that the simplex algorithm, starting from the basic feasible solution $\vec{x} = (0, 0, \ldots, 0, 1, 3, 7, \ldots, 2^m - 1)$, could visit all 2^m basic feasible solutions before finding the optimum.

[2] Note the limit in the sum in the objective function: only the first m of the $n = 2m$ variables are summed.