Algorithmic Combinatorics — Wi 2016–17

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Exercise Sheet 11

Due date: 14:00, Jan 24th, by the end of the lecture.
Late submissions will wish they were Ramos at the Sánchez Pizjuán.

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 In class we formulated the dual program for a linear program with upper bound constraints and non-negative variables. In this exercise you shall extend duality to all linear programs.

(a) A very general form for a linear program (P) is as follows:

$$\begin{array}{lrcrcl} \max & \vec{c}_1^{\,T}\vec{x}_1 & + & \vec{c}_2^{\,T}\vec{x}_2 & & \\ \text{subject to} & A^{(1,1)}\vec{x}_1 & + & A^{(1,2)}\vec{x}_2 & \leq & \vec{b}_1, \\ & A^{(2,1)}\vec{x}_1 & + & A^{(2,2)}\vec{x}_2 & \geq & \vec{b}_2, \\ & A^{(3,1)}\vec{x}_1 & + & A^{(3,2)}\vec{x}_2 & = & \vec{b}_3, \\ \text{and} & & & \vec{x}_1 & \geq & \vec{0}, \end{array}$$

where $\vec{c}_j, \vec{x}_j \in \mathbb{R}^{n_j}$ for $1 \leq j \leq 2$, $\vec{b}_i \in \mathbb{R}^{m_i}$ for $1 \leq i \leq 3$, and each $A^{(i,j)}$ is an $m_i \times n_j$ real-valued matrix.

By combining the constraints optimally to upper bound the objective function, find the dual program to (P).

(b) Show that by first converting (P) into the form where all constraints are upper bounds and all variables are non-negative, one again obtains the dual program you found in part (a).

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S11.html`.]

Exercise 2 In this exercise you will complete our proof of the Duality Theorem for linear programming, and prove what is known as complementary slackness.

(a) Recall that we are assuming the primal (P) is bounded and feasible, casting it in equational form, and using the simplex algorithm with Bland's rule to reach a final basis B corresponding to an optimal basic feasible solution. Prove that $\vec{y}^* = (\vec{c}_B^{\,T} A_B^{-1})^T$ is feasible for the dual (D), and that $\vec{b}^T\vec{y}^* = \vec{c}^T\vec{x}$.

(b) Prove that for *any* optimal solution $\vec{y}$ to the dual (D), we have $y_j > 0$ only if in the primal (P), the jth constraint is satisfied with equality in every optimal solution.

Exercise 3 The Duality Theorem states that when considering the feasibility and boundedness of the primal and dual programs, at most four of nine cases can occur. Give, with justification, (small) linear programs and their duals to show that each of these four cases does actually occur.

Exercise 4 Consider the following linear program.

$$\begin{array}{lrcrcrcl} \text{min} & 4x_1 & + & 100x_2 & + & 120x_3 & & \\ \text{subject to} & 15x_1 & - & x_2 & - & 3x_3 & \leq & 10, \\ & -x_1 & - & x_2 & - & x_3 & \leq & -1, \\ & & & & & \vec{x} & \geq & \vec{0}. \end{array}$$

Find its dual, and use it to find the optimal value of the original program.

Exercise 5 Show that the following three statements are equivalent (you do not have to prove any of them).

(i) For all $m, n \in \mathbb{N}$, $A \in \mathbb{R}^{m \times n}$ and $\vec{b} \in \mathbb{R}^m$, the system $A\vec{x} = \vec{b}$ has a non-negative solution if and only if every $\vec{y} \in \mathbb{R}^m$ with $\vec{y}^T A \geq \vec{0}^T$ also satisfies $\vec{y}^T \vec{b} \geq 0$.

(ii) For all $m, n \in \mathbb{N}$, $A \in \mathbb{R}^{m \times n}$ and $\vec{b} \in \mathbb{R}^m$, the system $A\vec{x} \leq \vec{b}$ has a non-negative solution if and only if every non-negative $\vec{y} \in \mathbb{R}^m$ with $\vec{y}^T A \geq \vec{0}^T$ also satisfies $\vec{y}^T \vec{b} \geq 0$.

(iii) For all $m, n \in \mathbb{N}$, $A \in \mathbb{R}^{m \times n}$ and $\vec{b} \in \mathbb{R}^m$, the system $A\vec{x} \leq \vec{b}$ has a solution if and only if every non-negative $\vec{y} \in \mathbb{R}^m$ with $\vec{y}^T A = \vec{0}^T$ also satisfies $\vec{y}^T \vec{b} \geq 0$.

Exercise 6 Use any of the statements from Exercise 5 to give an alternative proof of the Duality Theorem that avoids using the simplex algorithm.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S11.html`.]