Algorithmic Combinatorics — Wi 2016–17
Shagnik Das — Tibor Szabó

Exercise Sheet 12

Due date: 14:00, Jan 31st, by the end of the lecture.
Late submissions will be sent a bill for a wall.[1]

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Alice and Bob play a game. They each pick an integer in $\{1, 2, 3\}$. If the sum of their numbers is odd, Alice pays Bob €10, while if the sum is even, Bob gives Alice €10.[2]

(a) Identify all the (mixed) Nash equilibria. What is the value of the game?

(b) Alice plays an optimal strategy, but notices that Bob is choosing his number uniformly at random. How should she adjust her strategy to take advantage of his mistake?

(c) After Alice makes this change, Bob starts losing money quickly. Observing that there are only 4 ways he can win, but 5 ways for Alice to win, he suggests that he should only have to pay €8 when Alice wins.[10] Should Alice agree to these new terms?

---

[1] It's gonna be yuge!

[2] Having grown up with the likes of chess, cricket, Pokémon and Rock–Paper–Scissors, this so-called game may not seem to offer much in the way of entertainment to you, and certainly ought not to merit such high stakes. Are Alice and Bob foolish half-wits, you ponder, easily amused and parted from their money, or is there some deeper story behind the situation described above?

As always, your instincts are right on the money, so to speak.[3] Our protagonists have only very recently met each other, as they find themselves on a blind date[4] in the ridiculously opulent restaurant in Funchal, Madeira, called "Cris's".[5] After Alice entered the restaurant, the maître d'hôtel[6] showed her to the table where Bob was waiting, and she was impressed by his punctuality. However, after some brief introductions, and perfunctory remarks on how nice it was to get away from the cold weather, an uncomfortable silence descended upon the pair, broken only briefly by the waiter mercifully stopping by to take their orders.[7]

Eventually, aware that things were not going particularly well, Bob, an inveterate gambler, tried to break the tension with his zero-sum two-player game. "Say now, Alice," he said, with a gleam in his eye, "would you fancy playing a game?" After explaining the rules to her, he added, "you do know what odd and even numbers are, right?" He silently prided himself in remembering to ask, because most of the girls he dated literally could not even.

Alice was rather taken aback, for this was not the kind of proposition she had been expecting to face on a first date, and she was infuriated by his offensive question. She was tempted to storm off at that very moment, but the thought of the sumptuous espetada she had just ordered kept her in her seat. Instead, she decided to teach Bob a lesson by winning his money. "Why yes," she answered, voice cold as steel,[8] "I do believe I am familiar with arithmetic modulo two. Let the games begin."

And that, my friends, is where this game comes from.

Exercise 2 A father plays Rock-Paper-Scissors with his daughter.[11] However, there is a slight twist — while he can choose any of rock, paper or scissors, she is only allowed to choose rock or paper.[12] The rules otherwise remain the same: paper beats rock, rock beats scissors, and scissors beats paper. Determine the optimal strategies for each player in these conditions.

---

[3]Unless you thought there was no deeper story, in which case you were wrong. While you can be forgiven for this error, do try to be more correct when answering the exercises.

[4]This date had been set up by none other than Eve, a high-profile financial lawyer who was paid (extremely) well to travel the world and argue that her clients practised tax *avoidance*, not tax *evasion*. While trying to catch up on the latest movies during her last transatlantic flight, she was annoyed to find her neighbour — none other than our very own Bob — kept trying to make conversation. Her noise-cancelling headphones did help drown out his boring personal anecdotes, but his obliviousness to her obvious disinterest made Eve think Bob might be perfect for her old college roommate, Alice.

When Alice got a call from Eve, suggesting she meet this man named Bob, she was surprised, to say the least. Not only had Alice and Eve not spoken since their graduation, nearly a decade ago, but when they had spoken, it had rarely been on the best of terms. Indeed, Alice always felt somewhat uncomfortable with Eve around, fearing that she was always listening in on Alice's private conversations with her then-boyfriend, also named Bob. Still, Alice reasoned, that had been a long time ago, and as her job left her with little time to tend to her own personal life, what harm could come from meeting this man whose praises Eve was so enthusiastically singing?

[5]This opulence came as quite the shock to Bob. He had chosen Funchal because he had always wanted to visit the CR7 museum, and naturally assumed that Cris's was a restaurant owned by his hero, Cristiano Ronaldo. This was not the case, and it quickly became apparent to him that this would be a more expensive meal than he had bargained for.

[6]The maître d'hôtel, Pedro, though relatively young of age, was experienced at his craft, and was widely respected within the service industry. Coincidentally enough, he too had mistakenly chosen to work at Cris's in hope of one day meeting Ronaldo, but chose to stay after seeing the clientele this establishment attracted (and, more to the point, the size of the tips they offered).

[7]This silence may surprise you, given how chatty Bob was with Eve.[4] As it transpired, his earlier verbosity was not due to any conversational excellence, but rather to his extreme nervousness when flying.

[8]Unfortunately for Alice, the venom in her voice was wasted on Bob, who was utterly oblivious to her antagonism, and instead was delighted to have a chance to play a game. Having been sent to an all-boys boarding school, Bob had quickly developed the skills one tends to acquire in such an environment — he could belch most tunes on request, and had a healthy tolerance to dirt[9] — but was woefully inept when it came to understanding women.

[9]One often hears of draconian institutions where cleanliness is mandated, but Bob's school was not such a place. Hiding behind the mantra "boys will be boys," his teachers found it easier to turn a blind eye to the states of their charges' rooms. Perhaps this was for the best; in his first year, a lonely Bob's best friends were the family of four mushrooms that sprouted from the edge of the carpet at his bathroom's doorway.

[10]For all his other failings, Bob was a quick calculator.

[11]Almost unbelievably, this father–daughter duo also had the names Bob and Alice, but were not related to the couple from Exercise 1.

[12]When confronted about the unfair asymmetry of this game, Bob exclaims, "you think I set it up this way just so I could win? Ridiculous! Don't you know it's dangerous for children to play with scissors?" When asked if rocks made for safe playthings, Bob was no longer available for comment.

Exercise 3 In a non-zero-sum two-player game, each player has their own payoff matrix. If Alice plays strategy i and Bob plays strategy j, then Alice receives a payoff of $a_{i,j}$, while Bob receives a payoff of $b_{i,j}$. A pair of mixed strategies, $\vec{x}_0$ for Alice and $\vec{y}_0$ for Bob, is a (mixed) Nash equilibrium if neither player can improve their payoff by changing their strategy alone.[3] That is, for any other mixed strategies $\vec{x}$ and $\vec{y}$, $\vec{x}_0^T A\vec{y}_0 \geq \vec{x}^T A\vec{y}_0$ and $\vec{x}_0^T B\vec{y}_0 \geq \vec{x}_0^T B\vec{y}$.

Consider a game with the following payoff matrices.

$(a_{i,j}, b_{i,j})$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(4, 7)$	$(6, -1)$	$(1, 5)$
$i = 2$	$(6, 1)$	$(4, 5)$	$(-3, 0)$
$i = 3$	$(2, 3)$	$(5, 5)$	$(4, 1)$

(a) Show that neither player would ever play their third strategy in any Nash equilibrium.

(b) Show that there is no pure Nash equilibrium.[4]

(c) Find a mixed Nash equilibrium.

Exercise 4 If $\vec{G}$ is a directed graph with n vertices and m arcs, we define a matrix $A(\vec{G})$ as follows. $A(\vec{G})$ is an $n \times m$ matrix with rows corresponding to the vertices of $\vec{G}$, and columns corresponding to the arcs of $\vec{G}$. The entry $A(\vec{G})_{v,e}$ on row $v \in V(\vec{G})$ and column $e \in E(\vec{G})$ is 1 if the arc e starts at v, -1 if the arc e ends at v, and 0 otherwise.

(a) Suppose $\vec{C}$ is a directed graph whose underlying undirected graph is a cycle ($\vec{C}$ itself need not be a directed cycle). Prove that $\det(A(\vec{C})) = 0$.

(b) Prove that for any directed graph $\vec{G}$, $A(\vec{G})$ is totally unimodular.

(c) Deduce that a flow network with integer capacities has an integer-valued optimal flow.

Exercise 5 Let $\mathcal{D}$ be a finite collection of congruent closed disks in the plane, such that any two have a point in common. Show that $\mathcal{D}$ has a transversal of size 4.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S12.html`.]

Exercise 6 Let $\mathcal{J}$ be a finite family of d-intervals not containing three pairwise-disjoint d-intervals; that is, there are no $J_1, J_2, J_3 \in \mathcal{J}$ with $J_i \cap J_j = \emptyset$ for every $1 \leq i < j \leq 3$. Show that $\mathcal{J}$ has a transversal of size $4d^2$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S12.html`.]

[3] Note that, unlike in zero-sum two-player games, the players may be able to improve their payoffs if they *both* change their strategies together.

[4] That is, one where the players always play the same strategy.