Algorithmic Combinatorics — Wi 2016–17
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Exercise Sheet 12

Due date: 14:00, Jan 31st, by the end of the lecture.
Late submissions will be sent a bill for a wall.[1]

You should try to solve all of the exercises below, and submit two solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Alice and Bob play a game. They each pick an integer in $\{1, 2, 3\}$. If the sum of their numbers is odd, Alice pays Bob €10, while if the sum is even, Bob gives Alice €10.

(a) Identify all the (mixed) Nash equilibria. What is the value of the game?

(b) Alice plays an optimal strategy, but notices that Bob is choosing his number uniformly at random. How should she adjust her strategy to take advantage of his mistake?

(c) After Alice makes this change, Bob starts losing money quickly. Observing that there are only 4 ways he can win, but 5 ways for Alice to win, he suggests that he should only have to pay €8 when Alice wins. Should Alice agree to these new terms?

Exercise 2 A father plays Rock-Paper-Scissors with his daughter. However, there is a slight twist — while he can choose any of rock, paper or scissors, she is only allowed to choose rock or paper. The rules otherwise remain the same: paper beats rock, rock beats scissors, and scissors beats paper. Determine the optimal strategies for each player in these conditions.

[1] It's gonna be yuge!

Exercise 3 In a non-zero-sum two-player game, each player has their own payoff matrix. If Alice plays strategy i and Bob plays strategy j, then Alice receives a payoff of $a_{i,j}$, while Bob receives a payoff of $b_{i,j}$. A pair of mixed strategies, $\vec{x}_0$ for Alice and $\vec{y}_0$ for Bob, is a (mixed) Nash equilibrium if neither player can improve their payoff by changing their strategy alone.[2] That is, for any other mixed strategies $\vec{x}$ and $\vec{y}$, $\vec{x}_0^T A\vec{y}_0 \geq \vec{x}^T A\vec{y}_0$ and $\vec{x}_0^T B\vec{y}_0 \geq \vec{x}_0^T B\vec{y}$.

Consider a game with the following payoff matrices.

$(a_{i,j}, b_{i,j})$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(4, 7)$	$(6, -1)$	$(1, 5)$
$i = 2$	$(6, 1)$	$(4, 5)$	$(-3, 0)$
$i = 3$	$(2, 3)$	$(5, 5)$	$(4, 1)$

(a) Show that neither player would ever play their third strategy in any Nash equilibrium.

(b) Show that there is no pure Nash equilibrium.[3]

(c) Find a mixed Nash equilibrium.

Exercise 4 If $\vec{G}$ is a directed graph with n vertices and m arcs, we define a matrix $A(\vec{G})$ as follows. $A(\vec{G})$ is an $n \times m$ matrix with rows corresponding to the vertices of $\vec{G}$, and columns corresponding to the arcs of $\vec{G}$. The entry $A(\vec{G})_{v,e}$ on row $v \in V(\vec{G})$ and column $e \in E(\vec{G})$ is 1 if the arc e starts at v, -1 if the arc e ends at v, and 0 otherwise.

(a) Suppose $\vec{C}$ is a directed graph whose underlying undirected graph is a cycle ($\vec{C}$ itself need not be a directed cycle). Prove that $\det(A(\vec{C})) = 0$.

(b) Prove that for any directed graph $\vec{G}$, $A(\vec{G})$ is totally unimodular.

(c) Deduce that a flow network with integer capacities has an integer-valued optimal flow.

Exercise 5 Let $\mathcal{D}$ be a finite collection of congruent closed disks in the plane, such that any two have a point in common. Show that $\mathcal{D}$ has a transversal of size 4.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S12.html`.]

Exercise 6 Let $\mathcal{J}$ be a finite family of d-intervals not containing three pairwise-disjoint d-intervals; that is, there are no $J_1, J_2, J_3 \in \mathcal{J}$ with $J_i \cap J_j = \emptyset$ for every $1 \leq i < j \leq 3$. Show that $\mathcal{J}$ has a transversal of size $4d^2$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMII-2016-17/hints/S12.html`.]

[2] Note that, unlike in zero-sum two-player games, the players may be able to improve their payoffs if they *both* change their strategies together.

[3] That is, one where the players always play the same strategy.